M1 Maths

A6-3 Geometric Sequences

- recursive and explicit formulae
- sum to *n* terms
- sum to infinity

Summary **Learn** **Solve** **Revise** **Answers**

Summary

A geometric sequence is like an arithmetic sequence except that, in a geometric sequence, the ratio of successive terms is constant rather than the difference. If the first term is a and the ratio of successive terms is r, then $t_n = t_{n-1} \times r$.

The nth term is $a \times r^{n-1}$, the sum of the first n terms is $a\frac{1-r^n}{1-r}$ and the sum of all terms to infinity is $\frac{a}{1-r}$ provided $-1 < r < 1$.

Learn

Note that many of the concepts related to geometric sequences are similar to those related to arithmetic sequences. As these were covered in Module A6-2 (Arithmetic Sequences), some familiarity with them will be assumed here. If you haven't studied arithmetic sequences, it may be worth working through A6-2 before embarking on A6-3. Just about all school programs would cover arithmetic sequences before geometric sequences, anyway.

A geometric sequence (also called a *geometric progression* or *GP*) is one where the ratio of successive terms is always the same. Examples are: 5, 15, 45, 135, 405 and 32, 16, 8, 4, 2, 1, 0.5, 0.25.

It is tradition in a GP to call the first term a and the ratio of successive terms r. r is also called the common ratio. So, in the first example above, $a = 5$, $r = 3$; in the second example, $a = 32$, $r = ½$.

If a GP is defined recursively, then $t_n = t_{n-1} \times r$.
Defined explicitly, $t_n = a \times r^{n-1}$.
Make sure you can see why both of these are true.

Practice

Q1 Specify each of the following arithmetic sequences
(i) recursively using the t notation
(ii) recursively using a and r
(iii) explicitly.

(a) 2, 6, 18, 54
(b) 64, 16, 4, 1, ¼
(c) 1, −1, 1, −1, 1, −1, . . .
(d) 10, −11, 12.1, −13.31, . . .

Q2 List the first 4 terms of each of the following geometric sequences.

(a) $t_1 = 6$, $t_n = t_{n-1} \times 2$
(b) $t_1 = -4$, $t_n = t_{n-1} \times 3$
(c) $a = 34$, $d = -10$
(d) $a = 6$, $d = \frac{1}{2}$
(e) $t_n = {}^1/_{48} \times 4^n$
(f) $t_n = 10 \times (-1.2)^{n-1}$

Q3 For each of the following GPs, give the values of a and r, give an expression for the nth term and give the value of the 8^{th} term.

(a) 5, 10, 20, 40, . . .
(b) 3, 7.5, 18.75, 46.875, . . .
(c) 5, −5, 5, −5, . . .
(d) 64, −32, 16, −8, . . .

Q4 For these questions, you might need to write and solve simple equations using the explicit formula for the nth term.

(a) In the GP 8, 12, 18, . . ., which term is closest to 692?
(b) In the GP 20, 50, 125, . . ., how many terms are $<1\,000\,000$?
(c) In a GP, $t_7 = 4$, $t_{19} = 64$. Find a and r.
(d) In a GP, $t_5 = 80$, $t_{18} = 20.335$. Find t_{11}.
(e) In the GP with $a = 1$ and $r = 2$, if you multiply the 10^{th} and 24^{th} terms, you will get one of the other terms of the sequence. Which one?
(f) In the GP 16, 32, 64 . . ., if you multiply the 10^{th} and 24^{th} terms, then take the square root of the result, you will get one of the other terms of the sequence. Which one?

Geometric Series

A number of successive terms of a geometric sequence added together is called a geometric series. For example, 5 + 10 + 20 + 40 + 80 is a geometric series. Its value is 155. A geometric series is a sum of terms in a geometric sequence.

As with arithmetic series, there is a formula for the sum of the first n terms of a GP, S_n.

It is: $S_n = a\frac{1-r^n}{1-r}$ or $S_n = a\frac{r^n-1}{r-1}$

The first version is easier to use if $|r| < 1$; the second version is easier if $|r| > 1$. However, the first one will work in both situations and is the best one to remember. If $|r| > 1$, the top and bottom both become negative, but the negatives cancel out.

The derivation of the formulae is less obvious than for arithmetic series. It is as follows.

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

$$S_n = ar + ar^2 + ar^2 + ar^4 + \ldots + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$= a(1 - r^n)$$

$$Sn(1 - r) = a(1 - r^n)$$

$$S_n = a\frac{1-r^n}{1-r}$$

Multiplying top and bottom by –1, we get $S_n = a\frac{r^n-1}{r-1}$

Practice

Q5 Use the formulae to find the sums of the terms in the following geometric sequences.

(a) 6, 12, 24, 48, 96, 192, 384 (b) 100, 10, 1, . . . 0.0001, 0.00001

(c) The first 8 terms in the sequence with $a = 2$, $r = 3$

(d) The first 30 terms in the sequence with $a = 12.6$, $r = 0.9$

(e) The first 24 terms in the geometric sequence −20, 15, −11.25, 8.4375, . .

Q6 (a) Find the sum of the powers of 3 from 3^3 to 3^{12} inclusive

(b) Find $2 + 1 + {}^1/_2 + {}^1/_4 + {}^1/_8 + {}^1/_{16} + {}^1/_{32}$

Q7 Gervill earnt $54 400 in 2014. Each year he earns 6% more than the previous year. How much will he earn in total from 2014 to 2039 inclusive?

Q8 In which year will Gervill earn his millionth dollar (excluding anything he might have earnt before 2014)?

Q9 The geometric series $10 + 15 + 22.5 + 33.75 + \ldots + x$ is equal to 748.867.

(a) How many terms in the series?

(b) What is the value of x?

Q10 The 2nd term of a GP is 24; the 9th term is −0.89407. Find the 7th term.

Sum to Infinity of a Geometric Series

Consider the geometric series 1 + 2 + 4 + 8 + 16 + ... continued for ever, to infinity. If we add all the terms, going on for ever – an infinite number of terms – then we call this the sum to infinity of the series.

As $r > 1$, the terms become larger and larger in magnitude. The sum to infinity will obviously be infinite.

The same happens if $r < -1$, e.g. 1 – 2 + 4 – 8 + 16 – . . . The further we go, the larger the sum becomes and if we continue the series to infinity, the sum will be infinite.

However, if $-1 < r < 1$, then the terms become smaller and smaller and the sum of an infinite number of terms will be finite. So we can find the sum to infinity of a geometric series as long as $-1 < r < 1$.

The formula for the sum to infinity is easy. We know that $S_n = \frac{1-r^n}{1-r}$.

If n becomes extremely large, r^n becomes extremely small (as long as $-1 < r < 1$) and $a\frac{1-r^n}{1-r}$ gets very close to $a\frac{1}{1-r}$ or $\frac{a}{1-r}$.

So the sum to infinity of a geometric series is $\frac{a}{1-r}$ (as long as $-1 < r < 1$).

As an example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ ($a = 1$, $r = \frac{1}{2}$) $= \frac{1}{1-{}^1/_2} = 2$

And $4 - 1 + \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \ldots$ ($a = 4$, $r = -\frac{1}{4}$) $= \frac{4}{1+\frac{1}{4}} = 3\frac{1}{5}$

Practice

Q11 Find the sum to infinity (if possible) of each of the following geometric sequences.

(a) $a = 6$, $r = \frac{1}{2}$

(b) $a = -4$, $r = -\frac{1}{5}$

(c) 1, 0.9, 0.81, 0.729, . . .

(d) 4, 5, 6.25, . . .

(e) $a = \frac{1}{12}$, $r = -2$

(f) 0.5, −0.25, 0,125, . . .

Q12 Fat Harry goes on an exercise program, but gets lazy. He walks 10 km the first day. Each subsequent day he walks 10% less than the previous day.

(a) How far will he walk on the 20th day?

(b) How far will he walk in total in the first 20 days?

(c) Assuming he keeps this up for ever, how far will he walk altogether?

Q13 Bronson leaves the pub and walks 1 km east, ½ km south, 250 m west, 125 m north, 62.5 m east and so on in a spiral pattern. How far from the pub does he end up?

Q14 45 000 people attended a concert and paid $80 each. 20% of them then stayed for a second performance and paid another $80 each. 20% of those who stayed, then stayed for a third performance, paying another $80, and so on for ever. How much would have been paid in total?

Solve

Q51 A zombie wanted to walk to the supermarket 16 km away. The first day he walked 10 km. The next day he could only manage one third that distance. Each subsequent day he walked one third what he had the previous day. Did he ever reach the supermarket?

Q52 A swinging door is opened 90°. When let go, it swings back 150° (so it is 60° from the closed position), then it swings forward 120° (so it is 60° from the closed position), then back 96° (so it is 36° from the closed position) and so on, each swing being $^4/_5$ of the previous swing. How far from the closed position does it stop?

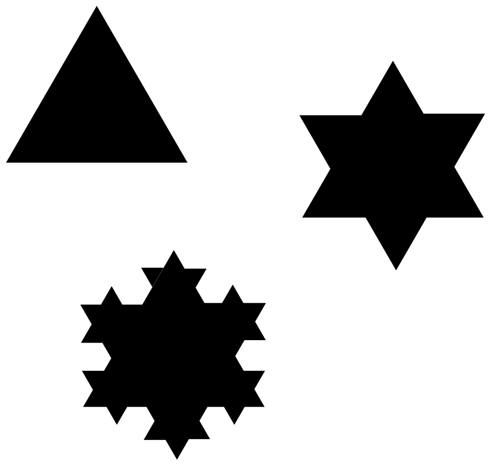
Q53 A ball is dropped from a height of 1 m. Each time it bounces, it rises to 80% of the previous height.

(a) How many times will it bounce?

(b) How far will it travel in total?

Q54 A Koch Snowflake is constructed like this:

- Draw an equilateral triangle with area 1 m^2.
- Replace the centre third of each side with two lines both the length of the original third.
- Replace the centre third of each side of the resulting figure with two lines both the length of the original third.
- Repeat *ad infinitum* (for ever).



Find the perimeter and area of the resulting shape.

Revise

Revision Set 1

Q61 Specify the following geometric sequences
(i) recursively using the t notation
(ii) recursively using a and d
(iii) explicitly.

(a) 5, 15, 45, 135, 505, . . . (b) 40, −20, 10, −5, 2.5, . . .

Q62 List the first 4 terms of each of the following sequences.

(a) $a = 8$, $r = 1.5$ (b) $a = -4$, $r = -1$

Q63 For each of the following geometric sequences, give the values of a and r, give an expression for the nth term and give the value of the 20th term.

(a) 40, 100, 250, 375, . . . (b) 12, − 8, $5\frac{1}{3}$, − $3\frac{5}{9}$, . . .

Q64 The 2nd term of a GP is $\frac{1}{8}$; the 11th term is −64. Find the 7th term and the sum of terms 1 to 12.

Q65 Find the sum of the powers of 1.2 from 1.2^2 to 1.2^{18} inclusive

Q66 Porky Pat goes on an exercise program, but has trouble keeping it up. He walks 4 km the first day. Each subsequent day he walks 20% less than the previous day.
(a) How far will he walk on the 18th day?
(b) How far will he walk in total in the first 18 days?
(c) Assuming he keeps this up for ever, how far will he walk altogether?

Revision Set 2

Coming soon

Revision Set 3

Coming soon

Answers

Q1 (a) (i) $t_1 = 2$, $t_n = t_{n-1} \times 3$ (ii) $a = 2$, $r = 3$ (iii) $t_n = 2 \times 3^n$

(b) (i) $t_1 = 64$, $t_n = t_{n-1} \times ¼$ (ii) $a = 64$, $r = ¼$ (iii) $t_n = 256 \div 4^n$ or $t_n = 64 \times ¼^{n-1}$

(c) (i) $t_1 = 1$, $t_n = t_{n-1} \times (-1)$ (ii) $a = 1$, $r = -1$ (iii) $t_n = (-1)^n$

(d) (i) $t_1 = 10$, $t_n = t_{n-1} \times -1.1$ (ii) $a = 10$, $r = -1.1$ (iii) $t_n = 10 \times 1.1^n$

Q2 (a) 6, 12, 24, 48 (b) −4, −12, −36, −108
(c) 48, 24, 12, 6 (d) 6, −0.6, 0.06, −0.006

(e) $^1/_{12}$, $^1/_3$, $^4/_3$, $^{16}/_3$ (f) 10, −12, 14.4, −1.728

Q3 (a) 5, 2, $5 \times 2^{n-1}$, 640 (b) 3, 2.5, $3 \times 2.5^{n-1}$, 1831.05
(c) 5, −1, $5 \times -1^{n-1}$, −5 (d) 64, −½, $64 \times -0.5^{n-1}$

Q4 (a) Term 12 (b) 12 (c) $a = 1, r = \sqrt[3]{2}$ (d) 42.52 (e) 33rd (f) 37th

Q5 (a) 762 (b) 111.11111 (c) 6560 (d) 120.66 (e) −11.44

Q6 (a) 797 148 (b) $1^{31}/_{32}$

Q7 $3 218 107.22
Q8 2026
Q9 (a) 9 (b) 256.289
Q10 −2.2888
Q11 (a) 12 (b) $-3^1/_3$ (c) 10 (d) ∞ (e) ∞ (f) 0.4
Q12 (a) 1.351 km (b)87.84 km (c) 100 km
Q13 666.7 m
Q14 $4 500 000

Q51 No Q52 6.667° Q53 (a) ∞ (b) 9 m Q54 Perimeter ∞, Area 1.6 m^2

Q61 (a) $t_1 = 5$, $t_n = t_{n-1} \times 3$, $a = 5, r = 3$, $t_n = 5 \times 3^{n-1}$
(b) $t_1 = 40$, $t_n = t_{n-1} \times -½$ $a = 40, r = -½$, $t_n = 40 \times (-½)^{n-1}$

Q62 (a) 8, 12, 18, 27 (b) −4, 4, −4, 4

Q63 (a) $a = 40, r = 2.5, t_n = 40 \times 2.5^{n-1}$, $t_{20} = 1\ 455\ 191\ 523$
(b) $a = 9, r = 2, t_n = 9 \times 2^{n-1}$, $t_{20} = –0.0054131…$
Q64 −4, 4095
Q65 152.54
Q66 (a) 72 m (b) 19.64 km (c) 20 km