M1 Maths

A6-2 Arithmetic Sequences

- recursive and explicit formulae
- sum to *n* terms

Summary Learn Solve Revise Answers

Summary

An arithmetic sequence is a sequence of numbers where the difference between successive terms is constant.

An arithmetic sequence can be specified recursively by giving the first term and each subsequent term in terms of the previous term, e.g. $t_1 = 5$ and $t_n = t_{n-1} + 2$, where t_n is the nth term.

In an arithmetic sequence, the first term is usually called a and the difference between successive terms is called d. So the sequence can also be specified by stating a and d. In the above case, $a = 5$ and $d = 2$.

Alternatively the sequence can be specified explicitly by giving a formula for the terms in terms of the term number. In the above case $t_n = 5 + (n - 1) \times 2$.

S_n, the sum of the first n terms of an arithmetic sequence is $^n/_2\,[2a + (n - 1)d]$.

Learn

What is an Arithmetic Sequence?

An arithmetic sequence is a sequence of numbers where the difference between successive numbers is always the same.

[Note that, being an adjective, *arithmetic* is pronounced with the stress on the *e* rather than on the first *i*. Note also that an arithmetic sequence is sometimes called an *arithmetic progression*, or *AP* for short.]

15, 17, 19, 21, 23, 25 is an arithmetic sequence. So is 7, 1.5, –4, –9.5, –15.

As with any sequence, the numbers in an arithmetic sequence are called *terms*. The first term is called t_1, the second term t_2 and so on; the nth term is called t_n.

We can specify a large, or infinite number of terms by placing . . . in the middle or at the end of a list of terms, e.g.

3, 7, 11, 15, 18 . . . 93, 96, 99

15, 20, 25, 30, 35 . . .

Recursive and Explicit Formulae

Consider the arithmetic sequence 11, 14, 17, 20, 23. We can specify this sequence with a recursive formula or with an explicit formula.

A recursive formula consists of the first term and the difference between successive terms. In this case, $t_1 = 11$, $t_n = t_{n-1} + 3$.

It is common with APs to call the first term a and the difference between successive terms d. So we could also specify the sequence recursively as $a = 11$, $d = 3$.

An explicit formula is a formula for the nth term in terms of n, the term number. In this case it would be $t_n = 8 + 3n$.

To get the explicit formula, we start with the first term, a, then, for the second term add d, for the third term, add $2d$ and so on, for the nth term, adding $(n-1)d$. So the explicit formula is always $t_n = a + (n-1)d$. In the example above, this is $t_n = 11 + 3(n-1)$, which simplifies to $t_n = 8 + 3n$.

Practice

Q1 Specify each of the following arithmetic sequences

(i) recursively using the t notation

(ii) recursively using a and d

(iii) explicitly.

(a) 7, 9, 11, 13, 15

(b) 32, 29, 26, 23, 20

(c) −5, −1, 3, 7, 11, . . .

(d) 2, 1, 0, −1, −2, . . .

Q2 List the first 4 terms of each of the following arithmetic sequences.

(a) $t_1 = 12$, $t_n = t_{n-1} + 5$

(b) $t_1 = -4$, $t_n = t_{n-1} + 2$

(c) $a = 34$, $d = -10$

(d) $a = 6$, $d = ½$

(e) $t_n = -5 + 3n$

(f) $t_n = 17.5 - 6.5n$

Q3 For each of the following APs, give the values of a and d, give an expression for the nth term and give the value of the 20th term.

(a) 9, 11, 13, 15, . . .

(b) 22, 29, 36, . . .

(c) 5, 3, 1, −1, . . .

(d) 64, 52, 40, 32, . . .

Q4 For these questions, you might need to write and solve simple equations using the explicit formula for the nth term.

(a) In the AP 8, 13, 18, . . ., what number term is 88?

(b) In the AP 31, 38, 45, . . ., how many terms are <500?

(c) In an AP, $t_6 = 4$, $t_{18} = 64$. Find a and d.

(d) In an AP, $t_5 = 80$, $t_{28} = 264$. Find t_{11}.

(e) In the AP with $a = 24$ and $d = 12$, if you add the 40th and 94th terms, you will get one of the other terms of the sequence. Which one?

(f) In the AP 20, 25, 30 . . ., if you add the 12th and 44th terms, then halve the result, you will get one of the other terms of the sequence. Which one?

Arithmetic Series

The sum of the terms in an arithmetic sequence is called an arithmetic series. For example, 5 + 8 + 11 + 14 + 17 is an arithmetic series. Its value is 55.

The word *series* is used a bit differently from in normal English. In normal English, a series consists of a number of things. A TV series consists of more than one program. In maths, an arithmetic series is just a single thing – the sum of the terms in an arithmetic sequence. By the way, the plural of *series* is *series* – a bit like fish or sheep.

We can work out the value of an arithmetic series like this. The sum is the average of the terms multiplied by the number of terms. The average of the terms is equal to the average of the first and last terms. [To see this, realise that the average of the second and second-last terms is the same as the average of the first and last, and so on.]

Suppose we want the sum of the first n terms in an arithmetic sequence. The first term will be a and the nth term (the last of the n terms) will be $a + (n - 1)d$. The average of these is $[a + a + (n - 1)d] \div 2$, i.e. $[2a + (n - 1)d] \div 2$. Multiplying by the number of terms gives us $^n/_2[2a + (n - 1)d]$.

We often call the sum of the first n terms of an AP S_n. It can also be called *the sum to n terms*.

$$S_n = {}^n/_2[2a + (n - 1)d].$$

It can be worth remembering this, though you should be able to work out the sums using *average* × *number of terms* without remembering the formula.

Practice

Q5 Use the formula to find the sums of the terms in the following arithmetic sequences.

(a) 5, 7, 9, 11, 13, 15, 17

(b) the first 25 terms of the arithmetic sequence 17, 15, 13, 11, . . .

(c) 10, 15, 20, . . . 70, 75, 80

(d) The first 20 terms in the sequence with $a = 20$, $d = 4$

(e) The first 30 terms in the sequence with $a = 41$, $d = -5$

(f) The first 100 terms in the arithmetic sequence $-20, -12, -4, 4, \ldots$

(g) The first 15 terms in the sequence where $t_n = 40 + 6n$

Q6 (a) Find the sum of the whole numbers from 21 to 74 inclusive

(b) Find the sum of the odd numbers from 1 to 99 inclusive

Q7 Grimbo wants to lay floor boards on a triangular deck. The first board needs to be 14 cm long and each board after that needs to be 6 cm longer than the previous one. The last board will be 548 cm long. How many metres of board should he buy?

Q8 The sum $10 + 15 + 20 + 25 + \ldots + x$ is equal to 2025.

(a) How many terms in the sequence?

(b) What is the value of x?

Q9 How many terms of the AP $-5, -3, -1, 1, 3, 5, \ldots$ must be added for the sum to exceed 1000?

Q10 The 4^{th} term of an AP is 31; the 17^{th} term is 122. Find the sum of the 4^{th} to the 17^{th} terms inclusive.

Solve

Q51 Show that the sum of the first n odd numbers is n^2.

Q52 In an AP, $t_5 + t_{12} = 30$ and $t_{22} - t_8 = 56$. Find the sum of the first 30 terms.

Q53 Each day in March and April, the local dragon ate 2 more villagers than it did the previous day. If it ate a total of 1581 villagers in March, how many did it eat in April?

Q54 In an AP, the sum of the first 10 terms is 265 and the sum of the next 10 terms is –35. Find the 4^{th} term.

Revise

Revision Set 1

Q61 Specify the following arithmetic sequence
(i) recursively using the t notation
(ii) recursively using a and d
(iii) explicitly.
8, 13, 18, 23, 28, . . .

Q62 List the first 4 terms of each of the following sequences.
(a) $a = 8$, $d = 2$ (b) $a = -4$, $d = -0.5$

Q63 For each of the following arithmetic sequences, give the values of a and d, give an expression for the nth term, the value of the 90^{th} term and the sum of the first 25 terms.
(a) 31, 22, 13, 4, −5, . . . (b) 9, 16, 23, 30, 37, . . .

Q64 The sum $63 + 66 + 69 + 72 + \ldots + x$ is equal to 1593.
(a) How many terms in the sequence? (b) What is the value of x?

Q65 The 4th term of an AP is 71; the 14th term is 161. Find the 11th term and the sum of terms 1 to 34.

Revision Set 2

Q71 Specify the following arithmetic sequence
(i) recursively using the t notation
(ii) recursively using a and d
(iii) explicitly.
21, 13, 5, −3 . . .

Q72 List the first 4 terms of each of the following sequences.
(a) $a = -4$, $d = ½$ (b) $a = 22$, $d = -3$

Q73 For each of the following arithmetic sequences, give the values of a and d, give an expression for the nth term, the value of the 40^{th} term and the sum of the first 20 terms.
(a) −22, −17, −12, . . . (b) 9.8, 11.2, 12.6, 14, . . .

Q74 The sum $12 + 18 + 24 + 30 + \ldots + x$ is equal to 1650.
(a) How many terms in the sequence? (b) What is the value of x?

Q75 The 7th term of an AP is 14; the 12th term is 59. Find the 50th term and the sum of terms 11 to 20 inclusive.

Revision Set 3

Coming soon

Answers

Q1 (a) (i) $t_1 = 7,\ t_n = t_{n-1} + 2$ (ii) $a = 7,\ d = 2$ (iii) $t_n = 5 + 2n$
(b) (i) $t_1 = 32,\ t_n = t_{n-1} - 3$ (ii) $a = 32,\ d = -3$ (iii) $t_n = 35 - 3n$
(c) (i) $t_1 = -5,\ t_n = t_{n-1} + 4$ (ii) $a = -5,\ d = 4$ (iii) $t_n = -9 + 4n$
(d) (i) $t_1 = 2,\ t_n = t_{n-1} - 1$ (ii) $a = 2,\ d = -1$ (iii) $t_n = 3 - n$

Q2 (a) 12, 17, 22, 27 (b) −4, −2, 0, 2
(c) 34, 24, 14, 4 (d) 6, 6½, 7, 7½
(e) −2, 1, 4, 7 (f) 11, 4.5, −2, −8.5

Q3 (a) 9, 2, $7 + 2n$, 47 (b) 22, 7, $15 + 7n$, 155
(c) 5, −2, $7 - 2n$, −33 (d) 64, −12, $76 - 12n$, −164

Q4 (a) 17 (b) 68 (c) −16, 5 (d) 128 (e) 135th (f) 59th

Q5 (a) 121 (b) −175 (c) 675 (d) 1160 (e) −960 (f) 37 600
(g) 1320

Q6 (a) 2565 (b) 2500
Q7 (a) 252.9 m
Q8 (a) 27 (b) 140
Q9 34
Q10 1071

Q52 1290
Q53 3360
Q54 31

Q61 (i) $t_1 = 8,\ t_n = t_{n-1} + 5$ (ii) $a = 8,\ d = 5$ (iii) $t_n = 3 + 5n$
Q62 (a) 8, 10, 12, 14 (b) −4, −4.5, −5, −5.5
Q63 (a) $a = 31,\ d = -9,\ t_n = 40 - 9n,\ t_{90} = -770,\ S_{25} = -1925$
(b) $a = 9,\ d = 7,\ t_n = 9 + 7(n - 1)$, $t_{90} = 632,\ S_{25} = 2325$
Q64 (a) 18 (b) 114
Q65 134, 7922

Q71 (i) $t_1 = 21,\ t_n = t_{n-1} - 8$ (ii) $a = 21,\ d = -8$ (iii) $t_n = 29 - 8n$
Q72 (a) –4, –3½, –3, –2½ (b) 22, 19, 16, 13
Q73 (a) $a = 22,\ d = 5,\ t_n = -27 + 5n,\ t_{40} = 173,\ S_{20} = 510$
(b) $a = 9.8,\ d = 1.4,\ t_n = 8.4 + 1.4n,\ t_{40} = 64.4,\ S_{20} = 462$
Q74 (a) 22 (b) 138
Q75 401, 905