M1 Maths

A5-9 Algebraic Transformations

- how changes to formulae affect their graphs and vice versa

Summary   Lead-In   Learn   Solve   Revise   Answers

Summary

Starting with the graph of $y = f(x)$,

$y = f(x) + a$ is the same graph but translated a units upwards (down if a is negative)

$y = a \times f(x)$ is the same but dilated by a factor of a from the x-axis

$y = f(x + a)$ is the same but translated a units to the left

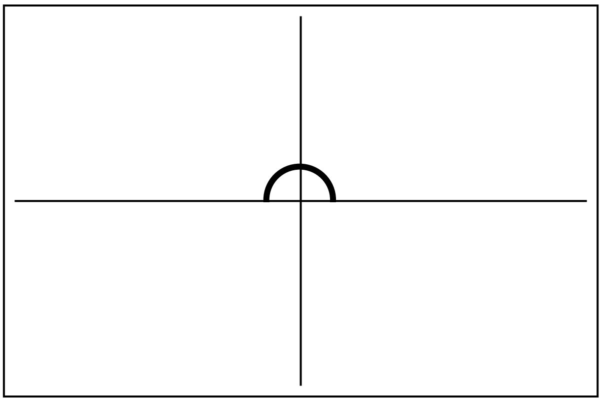
$y = f(a \times x)$ is the same but dilated by a factor of $^1/_a$ from the y-axis.

If more than one transformation is involved, e.g. with $y = 3\,f(2x - 5) + 1$, the graph is transformed successively, using the same order of operations as in the formula.

Lead-In

This lead-in activity will take a little while, but it's important to do it if you want algebraic transformations to seem easy and natural after you have learnt them.

Put the function $y = \sqrt{1 - x^2}$ into Y1 on your graphics calculator and draw it. Set the view window to $-7 \le x \le 7$ and $-5 \le y \le 5$. You should see half a circle centred on the origin with radius 1.



1. Now in Y2, put $y = \sqrt{1-x^2} + 1$ and draw Y1 and Y2 together. Then do likewise with $y = \sqrt{1-x^2} + 3$ and then $y = \sqrt{1-x^2} - 2$.

   Try to predict what $y = \sqrt{1-x^2} + 2$ and $y = \sqrt{1-x^2} - 4$ will look like. Check.

   Summarise your findings with a general rule about how the graph of $y = \sqrt{1-x^2} + c$ differs from the graph of $y = \sqrt{1-x^2}$.

2. Use a similar method to find out how the graph of $y = \sqrt{1-(x+c)^2}$ differs from the graph of $y = \sqrt{1-x^2}$. Use positive and negative values of *c*. Summarise your findings.

3. Use a similar method to find out how the graph of $y = c\sqrt{1-x^2}$ differs from the graph of $y = \sqrt{1-x^2}$. Use positive, negative and fractional values of *c*. Summarise your findings.

4. Use a similar method to find out how the graph of $y = \sqrt{1-(cx)^2}$ differs from the graph of $y = \sqrt{1-x^2}$. Use positive, negative and fractional values of *c*. Summarise your findings.

Learn

If we change the formula for a function, the graph will change. Such changes are called transformations. *Transformation* is just a fancy word for *change*.

There are four transformations you need to know about. They correspond to the four parts of the Lead-In.

1. If we transform $y = f(x)$ to $y = f(x) + c$, i.e. add *c* to the function, the graph moves upwards by *c* units. (Of course, if *c* is negative, it moves down.) This will actually work for any function, not just $\boldsymbol{y} = \sqrt{1-x^2}$.

   The reason for this should be fairly easy to see. If we transform $y = f(x)$ to $y = f(x) + 2$, for any *x*-value, the *y*-value is increased by 2, so that point on the graph moves up 2 units.

2. If we transform $y = f(x)$ to $y = f(x + c)$, i.e. add *c* to the independent variable, the graph moves to the left by *c* units. (Of course, if *c* is negative, it moves to the right.) This too will work for any function.

   At first sight, this might seem a little counter-intuitive and the reason is a bit harder to see. If we transform $y = f(x)$ to $y = f(x + 2)$, we get the same y-value for an x-value 2 less than in $y = f(x)$. So the graph moves 2 units to the left.

3. If we transform $y = f(x)$ to $y = cf(x)$, i.e. multiply the function by c, the graph is stretched vertically from the x-axis by a factor of c. In other words, every point on the graph becomes c times as far from the x-axis or the graph is dilated by a factor of c from the x-axis. If $c = 3$, the point (5, 2) moves to (5, 6) and (–1, –3) moves to (–1, –9). This too will work for any function. If c is negative, points above the x-axis move below the x-axis and vice versa. So the graph of $y = -cf(x)$ is a reflection in the x-axis of the graph of $y = cf(x)$. If $-1 < c < 1$, the graph will shrink towards the x-axis rather than being stretched away from it.

   Like No. 1, the reason for this one is fairly easy to see. If we transform $y = f(x)$ to $y = 2f(x)$, for any x-value, the y-value is 2 times as much; so every point on the graph moves to a new point 2 times as far from the x-axis, so the graph is dilated from the x-axis by a factor of 2.

4. If we transform $y = f(x)$ to $y = f(cx)$, i.e. multiply the independent variable by c, the graph is compressed horizontally towards the y-axis by a factor of c. In other words every point on the graph becomes $^1/_c$ times as far from the y-axis or the graph is dilated by a factor of $^1/_c$ from the y-axis. If $c = 3$, the point (6, 2) moves to (2, 2) and (–3, –3) moves to (–1, –3). This too will work for any function. If c is negative, points to right of the y-axis move to the left of the y-axis and vice versa. So the graph of $y = f(-cx)$ is a reflection in the y-axis of the graph of $y = f(cx)$. If $-1 < c < 1$, the graph will stretch from the y-axis rather than being compressed towards it.

   The reason for this is similar to the reason for No. 2. If we transform $y = f(x)$ to $y = f(2x)$, then for any y-value, the x-value only needs to be ½ as much to get that same y-value. So the graph is squashed towards the y-axis or dilated by a factor of ½ from the y-axis.

Make sure you know the above well, including understanding the reasons. If you do, you won't forget the effects of the four types of transformation.

Practice

Q1 Without using your calculator, draws graphs of the following functions. Draw them all on the same set of axes, but label them. Use your calculator to check your answers.

(a) $y = \sqrt{1-x^2}$ (b) $y = \sqrt{1-x^2} + 5$

(c) $y = \sqrt{1-x^2} - 1$ (d) $y = \sqrt{1-x^2} + 2$

(e) $y = \sqrt{1-(x-3)^2}$ (f) $y = \sqrt{1-(x+2)^2}$

(g) $y = \sqrt{1-(x-1)^2}$ (h) $y = \sqrt{1-(x+5)^2}$

Q2 Without using your calculator, draws graphs of the following functions. Draw them all on the same set of axes, but label them. Use your calculator to check your answers.

(a) $y = \sqrt{1-x^2}$ (b) $y = 2\sqrt{1-x^2}$

(c) $y = \frac{1}{2}\sqrt{1-x^2}$ (d) $y = 5\sqrt{1-x^2}$

(e) $y = -\sqrt{1-x^2}$ (f) $y = -3\sqrt{1-x^2}$

(g) $y = -\frac{1}{4}\sqrt{1-x^2}$ (h) $y = 0.1\sqrt{1-x^2}$

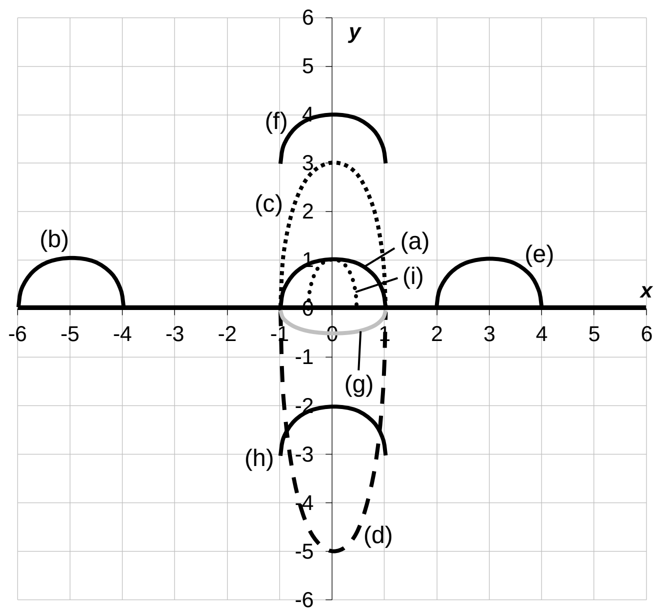
Q3 Without using your calculator, draws graphs of the following functions. Draw them all on the same set of axes, but label them. Use your calculator to check your answers.

(a) $y = \sqrt{1-x^2}$ (b) $y = \sqrt{1-(2x)^2}$

(c) $y = \sqrt{1-(\frac{1}{4}x)^2}$ (d) $y = \sqrt{1-(-3x)^2}$

(e) $y = \sqrt{1-(-\frac{1}{4}x)^2}$ (f) $y = \sqrt{1-(8x)^2}$

Q4 Write a formula for each of the functions shown in this diagram. Use your calculator to check that your functions match the pictures.



To help fix these ideas in your mind, try to do the next practice set just out of your head.

Practice

Q5 Explain why transforming $y = f(x)$ to $y = f(x) + c$ moves the graph c units upwards.

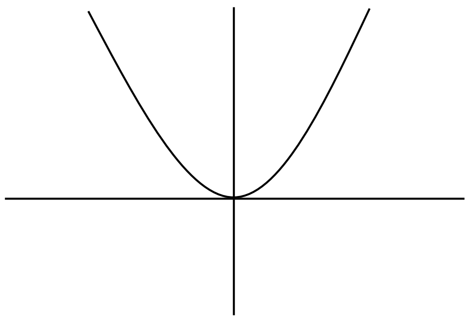
Q6 Explain why transforming $y = f(x)$ to $y = f(x + c)$ moves the graph c units to the left.

Q7 Explain why transforming $y = f(x)$ to $y = cf(x)$ dilates the graph vertically from the x-axis by a factor of c.

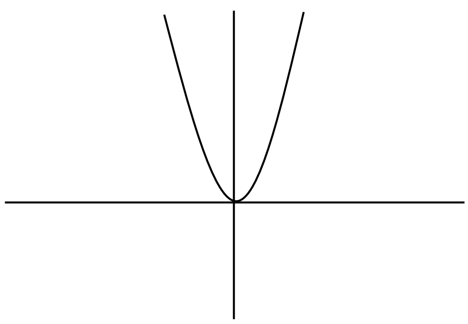
Q8 Explain why transforming $y = f(x)$ to $y = f(cx)$, dilates the graph horizontally from the y-axis by a factor of $^1/_c$.

The function $y = \sqrt{1-x^2}$ was chosen for the above because its graph is completely contained in a small area and the curve has ends. Its domain is $-1 \le x \le 1$ and its range is $0 \le x \le 1$. Because of this it is easy to see changes in height and width.

Most functions, however, have an infinite domain, so changes in height and width are less obvious. The graph of $y = x^2$ looks like this:



Stretching this vertically and compressing it horizontally will both result in this:



If seen as stretching vertically by a factor of 4, the formula would be $y = 4x^2$. If seen as compressing horizontally by a factor of 2, the formula would be $y = (2x)^2$. This of course simplifies to $y = 4x^2$. So the two transformations are in fact the same.

Multiple Transformation

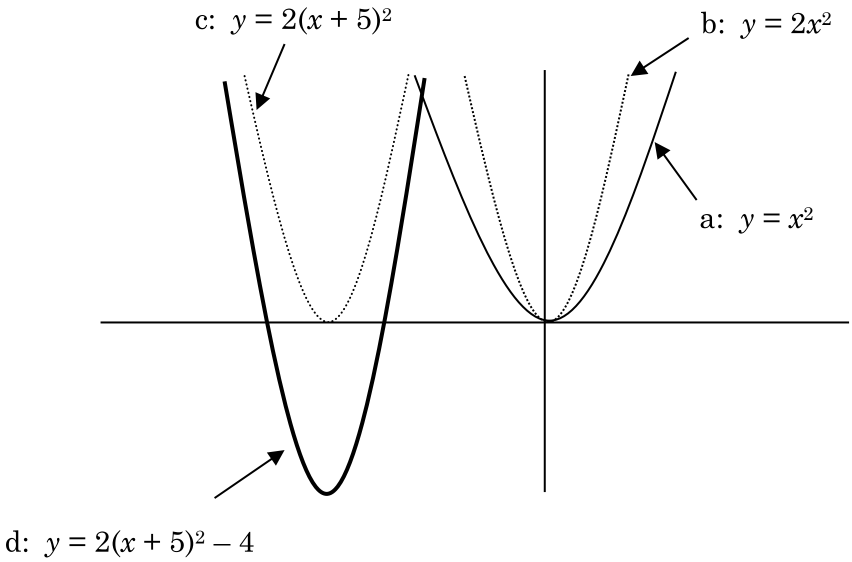
It is possible to follow one transformation with another.

Finding graphs from formulae

Suppose we start with $y = x^2$, then multiply the function by 2 to get $y = 2x^2$, then add 5 to the independent variable to get $y = 2(x + 5)^2$, then subtract 4 from the function to get $y = 2(x + 5)^2 - 4$.

$$y = x^2 \rightarrow y = 2x^2 \rightarrow y = 2(x + 5)^2 \rightarrow y = 2(x + 5)^2 - 4$$

The graph would be changed from a to b to c to d like this:

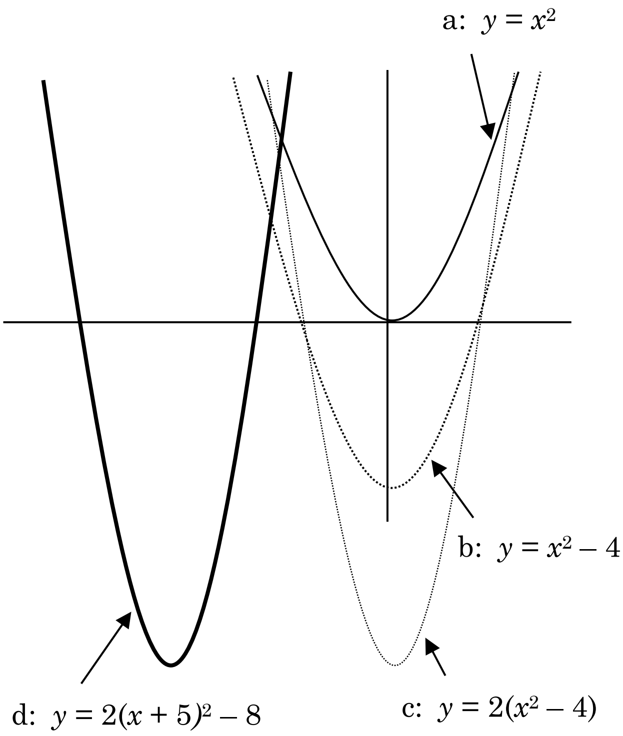


So, if we have a transformed formula like $y = 2(x + 5)^2 - 4$ and we want to find the transformed graph, first we determine what basic formula we started with (one whose graph we know), in this case $y = x^2$. Then we work out what transformations were applied and in what order: multiplying the function by 2; adding 5 to the independent variable; subtracting 4 from the function. Then we apply the same transformations to the basic graph in that same order.

Of course, in the case of $y = 2(x + 5)^2 - 4$, we could have added the 5 before multiplying by 2. This would lead to the same graph. But we couldn't subtract the 4 before multiplying by 2, because this would give us the formula

$y = x^2 \rightarrow y = x^2 - 4 \rightarrow y = 2(x^2 - 4) \rightarrow y = 2((x + 5)^2 - 4)$ which is $y = 2(x + 5)^2 - 8$

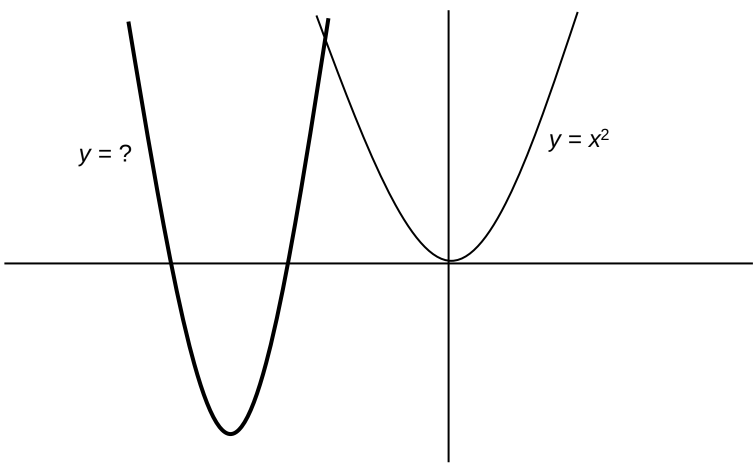
The graph for this formula is different.



We have to apply the transformations to the graph in the order in which the operations in the formula were applied to the independent variable, following the rules of order of operations. As long as we do this, the graph will come out right.

Finding formulae from graphs

If we know what the graph of $y = x^2$ looks like and we want to find the formula for the graph $y = ?$ below,



we would need to decide what sequence of transformations would turn the known graph into the unknown graph. It could be:

1. stretch by a factor of 2 vertically,
2. move 4 down

3. move 5 to the left

(Of course other sequences would work too.)

The corresponding changes to the formula would be

1.	stretch by a factor of 2 vertically,	multiply the function by 2
2.	move 4 down	subtract 4 from the function
3.	move 5 to the left	add 5 to x

So the formula would be transformed like this:

$$y = x^2 \rightarrow y = 2x^2 \rightarrow y = 2x^2 - 4 \rightarrow y = 2(x+5)^2 - 4$$

So the formula for the $y = ?$ graph is $y = 2(x+5)^2 - 4$

In reality, estimating some of the transformations from the graph will be a fairly approximate affair unless we know the coordinates of some points on the graphs.

Practice

Q9 Using your knowledge of the graphs of $y = \sqrt{1-x^2}$, sketch the graphs of the following functions without your calculator. Feel free to include some intermediate steps in your sketch. Use your calculator to check your answers.

(a) $y = 2\sqrt{1-x^2} - 3$ (b) $y = \sqrt{1-(x-3)^2} + 1$

(c) $y = \sqrt{1-(2x)^2} - 2$ (d) $y = -3\sqrt{1-(x+4)^2}$

(e) $y = \frac{1}{2}\sqrt{1-x^2} + 4$ (f) $y = -\sqrt{1-(\frac{1}{4}x)^2}$

(g) $y = 2\sqrt{1-(3x)^2} + 2$ (h) $y = 0.2\sqrt{1-(3(x+1))^2}$

Q10 Using your knowledge of the graphs of $y = x^2$, sketch the graphs of the following functions without your calculator. Feel free to include some intermediate steps in your sketch. Use your calculator to check your answers.

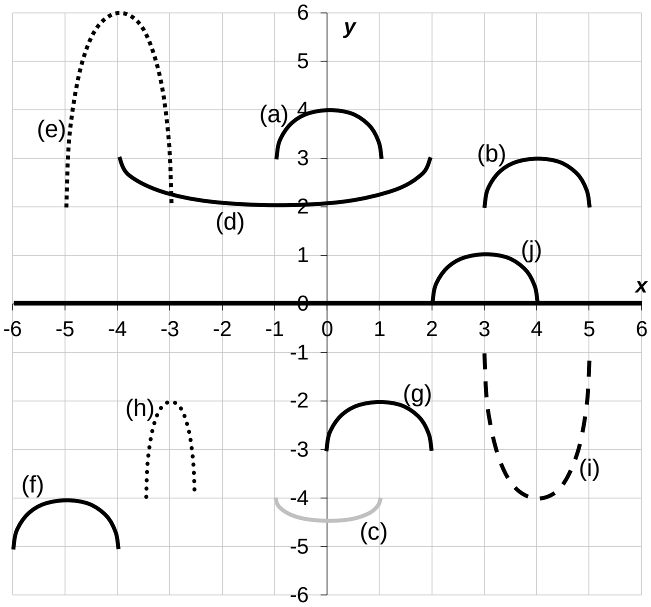
(a) $y = 2x^2 - 3$ (b) $y = \frac{1}{2}(x+1)^2$

(c) $y = 5(x-4)^2 + 1$ (d) $y = -2(x+1)^2 + 3$

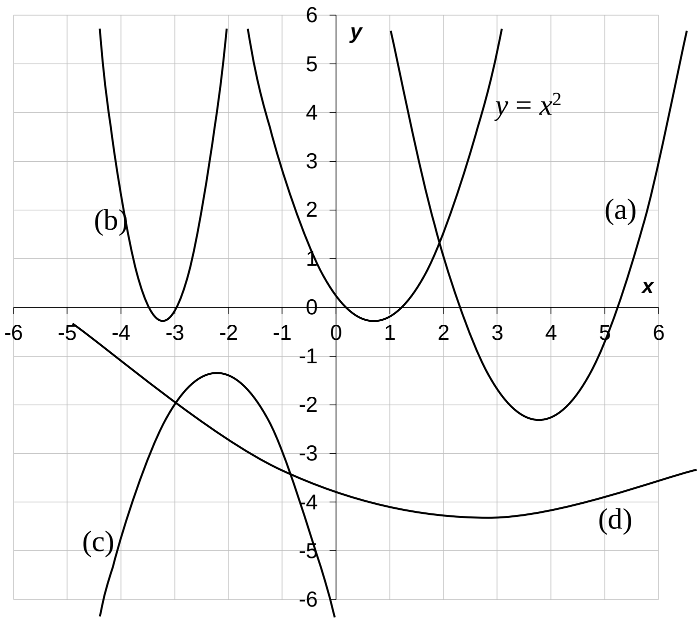
(e) $y = -\frac{1}{4}(2x)^2 - 5$ (f) $y = 4(2(x-3))^2$

(g) $y = 2(4(x-1))^2 - 5$ (h) $y = 0.1(-3(x+4))^2 - 2$

Q11 Using your knowledge of the graph of $y = \sqrt{1-x^2}$, find a formula for each of the following graphs. Feel free to include some intermediate steps in your calculation. Use your calculator to check your answers.



Q12 Using your knowledge of the graph of $y = x^2$, find an approximate formula for each of the following graphs. Use your calculator to check.

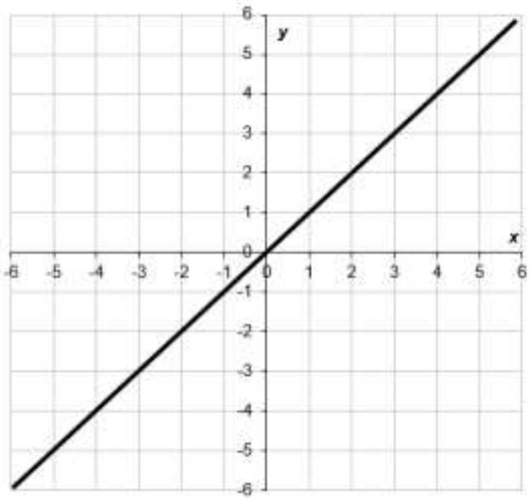


Function families, general form and the effects of the parameters on the shape of the graph.

You might have already learnt a few families of functions, their general form and the effects of the parameters on the shape of the graphs. You can now interpret the effects of the parameters on the graph in terms of transformations.

Take linear functions for example. The general form of a linear function is $y = mx + c$.

We can think of $y = x$ as the basic linear function. It looks like this:



It has a gradient of 1 and a y-intercept of 0.

If we multiply the function by m to get $y = mx$, then the graph is stretched vertically from the x-axis by a factor of m. This changes the gradient from 1 to m.

If we then add c to the function to get $y = mx + c$, then the graph is moved up c units. Thus the y-intercept changes from 0 to c.

Similar ideas can be applied to reciprocal functions of the general form $y = \frac{k}{x}$ and exponential functions of the form $y = ab^x$.

Consider quadratic functions. We can think of $y = x^2$ as the basic quadratic function. Other quadratic functions can be produced by introducing parameters representing transformations to get a general form $y = a(x + b)^2 + c$. Unfortunately, we normally use a different set of parameters in the standard form of a quadratic function which is $y = ax^2 + bx + c$. By completing the square, however, it is possible to change between the standard form $y = ax^2 + bx + c$ and the $y = a(x + b)^2 + c$ form (sometimes called the turning point form).

Practice

Q13 Graph $y = \sqrt{x}$ on your calculator, then sketch the graph of $y = \frac{1}{2}\sqrt{3x + 1} - 2$. Use your calculator to check your sketch.

Q14 Graph $y = \sin x$ on your calculator, then sketch the graph of $y = 2\sin 4(x - 30) + 3$. Use your calculator to check your sketch.

Q15 Rewrite the quadratic formula $y = x^2 - 6x + 5$ in the form $y = (x + a)^2 + b$. Then use that to sketch the graph of the function. Use your calculator to check your sketch.

Solve

Q51 Given that the formula for a circle of radius 1 centred on the origin is $x^2 + y^2 = 1$, find the formula for an ellipse with its long axis parallel to the x-axis and with domain $2 \le x \le 8$ and range $6 \le y \le 8$.

Q52 Find the radius and coordinates of the centre of the circle
$x^2 + y^2 - 8x + 4y - 5 = 0$

Revise

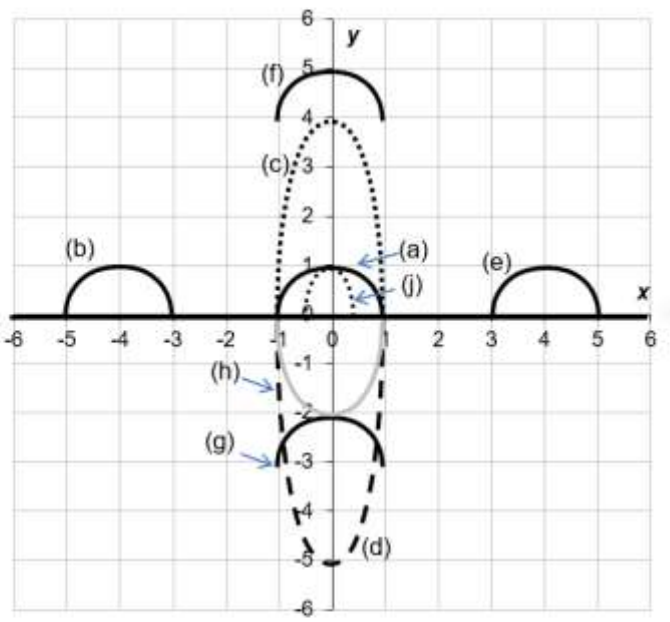
Revision Set 1

Q61 Without using your calculator, draws graphs of the following functions. Draw them all on the same set of axes, but label them. Use your calculator to check your answers.

(a) $y = \sqrt{1-x^2}$ (b) $y = \sqrt{1-x^2} - 3$ (c) $y = -4\sqrt{1-x^2}$

(d) $y = \sqrt{1-(x+2)^2}$ (e) $y = \sqrt{1-(3x)^2}$

Q62 Write an equation for each of the functions shown in this diagram. Use your calculator to check that your functions match the pictures.

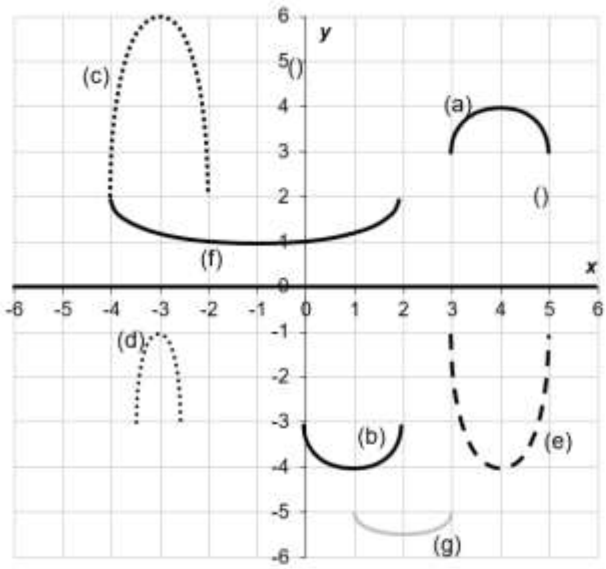


Q63 Using your knowledge of the graphs of $y = \sqrt{1-x^2}$, sketch the graphs of the following functions without your calculator. Feel free to include some intermediate steps in your sketch. Use your calculator to check your answers.

(a) $y = 3\sqrt{1-x^2} - 1$ (b) $y = \sqrt{1-(x-3)^2} + 2$

(c) $y = 0.5\sqrt{1-(3(x+1))^2}$ (d) $y = -\tfrac{1}{2}\sqrt{1-(\tfrac{1}{4}x)^2}$

Q64 Using your knowledge of the graphs of $y = \sqrt{1-x^2}$, find a formula for each of these graphs. Feel free to include some intermediate steps in your calculation. Use your calculator to check your answers.



Answers

Most answers should be checked with a graphics calculator.

Q51 $\left(\frac{x-5}{3}\right)^2 + (y-7)^2 = 1$

Q52 5, (4, –2)