M1 Maths

A5-6 Finding Formulae for Functions

- finding formulae for functions from known value pairs

Summary Learn Solve Revise Answers

Summary

A function in general form has a number of parameters. If we have a value pair for the function, we can sub the values into the general form of the function and get an equation with the parameters as unknowns.

If we have the same number of value pairs as parameters, then we can write the same number of equations as there are unknowns and we can solve the equations simultaneously to find the parameters and hence the particular function.

Learn

Linear Functions

Given gradient and a value pair

In Module A3-8 we saw how to find formulae for linear functions if we know the gradient and a value pair (i.e. a point the line passes through). Just to recap: the general form for a linear function is $y = mx + c$, where the parameter m is the gradient and the parameter c is the y-intercept. If the gradient is 2, then the formula is $y = 2x + c$. If the line passes through (3, 7), then we can sub these values for x and y to get an equation in c, and then solve the equation.

$$y = 2x + c$$
$$7 = 2 \times 3 + c$$
$$7 = 6 + c$$
$$c = 1$$

So the formula is $y = 2x + 1$.

Practice

Q1 Find the formulae for linear functions with the following gradients passing through the given points.

(a) gradient = 2, passing through (1, 4)

(b) gradient = –3, passing through (–1, –6)

(c) gradient = ½, passing through (–3, 4)

Q2 The fare for a taxi increases at $3 per kilometre. The fare for a 7 km trip is $26. Find the relation between fare and distance.

Given two value pairs

Suppose we have two value pairs for the function rather than the gradient and one value pair. For instance, suppose we need to find the formula for the linear function for which $y = 11$ when $x = 2$ and $y = 3$ when $x = 4$, i.e. the function whose graph passes through (2, 11) and (4, 3).

The general form is $y = mx + c$. Subbing in the first value pair, we get

$$11 = 2m + c.$$

Subbing in the second value pair, we get

$$3 = 4m + c.$$

Now that we can solve simultaneous equations, we can solve these two equations together to get $m = -4$ and $c = 19$. So our formula is $y = -4x + 19$.

Practice

Q3 Find the formulae for the linear functions passing through

(a) (1, 4) and (3, 8)

(b) (–1, 0) and (4, –10)

(c) (1, –5) and (3, 7)

(d) (–2, 4) and (3, –5)

Q4 A plumber charges a flat call-out fee plus so much per hour. The charge for a 3-hour job is $250 and the charge for a 5-hour job is $360. Find

(i) the call out fee

(ii) the rate per hour

(iii) how much she would charge for a 9½-hour job.

Q5 For undertakers at Drac's Funeral Parlour, there is a linear relation between the number of years experience and the pay per hour. Vlad has 9 years experience and earns $11.40 per hour; Blade has 17 years experience and earns $13.80 per hour.

Find:

(i) the relation between number of years experience and pay

(ii) how much Jezebel, who has 129 years experience, earns.

(iii) how many years experience Drusilla has if she earns $19.50 per hour.

Quadratic Functions

Using simultaneous equations allows us to find formulae for more complex functions.

Suppose we wanted the formula for a quadratic function. A quadratic function has the general form $y = ax^2 + bx + c$ with three parameters a, b and c. To find the three parameters, we need to solve three equations with the parameters as the unknowns. These three equations can come from three value pairs. So, if we have three value pairs, we can find the function.

Suppose the three value pairs are (–1, 5), (2, 6), (3, 9), i.e. the graph of the function passes through those three points.

Subbing the three value pairs into $y = ax^2 + bx + c$ gives us three equations in a, b and c:

$5 = a - b + c$
$6 = 4a + 2b + c$
$9 = 9a + 3b + c$

Then we can solve these simultaneously to get $a = 1$, $b = -2$, $c = 6$.

So our formula is $y = x^2 - 2x + 6$.

If you're hazy on solving simultaneous equations, go back and review them in Modules A4-3 and A5-5.

Practice

Q6 Find the formulae for quadratic functions which pass through these points.

(a) (–4, 3), (1, –3), (2, 3),
(b) (1, 4), (2, 9), (4, 31)
(c) (–1, –5), (1, 5), (2, 7)
(d) (–3, –27), (0, –6), (1, –7)

You would have noticed from Q5 (d), that if one of the value pairs has 0 for the x-value, it makes the equation solving process a bit easier.

Q7 The height of a cannonball in flight it a quadratic function of time. Find that function if the ball's height at different times is given by

t	0	2	5	8
h	60	150	210	180

You hopefully realised in Q7, that you only need three value pairs to find any quadratic: the fourth one is superfluous. Hopefully you used the one where $t = 0$ to make the equation solving easier.

Q8 Find a formula for the nth term in this sequence. The fact that the second differences are constant means that the formula will be a quadratic.

1, 3, 3, 1, –3, –9, –17

Other Polynomial Functions

A cubic has four parameters and so requires four equations to find these and therefore four value pairs. Similarly a quartic has five parameters and so on.

Solving four or five equations simultaneously by hand, though, is tedious and it is enough here just to know that it is possible. In Module A5-9, you will learn how to solve sets of simultaneous equations using the equation solver function on a graphics calculator, so you can do some then.

A corollary of the above is that exactly one cubic function can be drawn through any four points (as long as they all have different x-values), and so on.

Power Functions

Formulae for other types of functions can be found in the same way. The number of parameters determines the number of value pairs needed.

The power function $y = ax^n$ has two parameters, a and n. So two value pairs are required to find a power function

Suppose we have this information:

x	y
4	320
6	1080

Our equations will be

$$a \times 4^n = 320$$

$$a \times 6^n = 1080$$

We can solve these by dividing the second by the first. This gives us

$$\frac{a \times 6^n}{a \times 4^n} = \frac{1080}{320}$$

$$1.5^n = 3.375$$

$$n = \log_{1.5} 3.375$$

$n = \frac{\log 3.375}{\log 1.5} = 3$

Then we can sub this into either equation, e.g.

$a \times 4^3 = 320$

$a \times 64 = 320$

$a = 5$

So the function is $y = 5x^3$.

Practice

Q9 The rate, p, in Watts, with which a body radiates heat is a power function of its absolute temperature, T. A chicken radiates at 12.0 Watts when walking around the kitchen at $T = 310$ K and at 48.7 Watts when it comes out of the oven at $T = 440$ K.

(a) Find the relation between p and T for the chicken.

(b) Find p after the chicken has been in the fridge for 2 days at $T = 276$ K.

(c) At what temperature would the chicken radiate the same power as a 1 kilowatt electric heater?

Apologies to chicken lovers.

Q10 The rate at which a hot body cools by convection is a power function of the difference between the temperature of the body and the temperature of its surroundings. At 10° above the surrounding temperature, the body cools at 1.20° per minute. At 30° above the surrounding temperature, the body cools at 4.74° per minute.

(a) Find the relation between rate of cooling and temperature difference.

(b) Find the rate of cooling when the temperature difference is 6°

(c) Find the temperature difference if the body cools at 3° per minute.

Exponential Functions

In an exponential function of the form $A = Pe^{rt}$, A and t are the variables and P and r are parameters (e is the number 2.718281828…). P is always the initial value of A (the value of A when $t = 0$, sometimes written A_0 instead of P); a or r represents the growth rate.

As there are two parameters, we need two value pairs to define the function.

Suppose the population of a small city had grown exponentially since it was declared a city. 5 years after it was declared, the population was 12 640 and 11 years after it was declared, the population was 14 923. We need to find the relation between population and time since declaration.

The formula will be $A = Pe^{rt}$, where A is the population and t is the number of years since declaration. Our two value pairs are (5, 12 640) and (11, 14 923). Subbing these in, we get:

$$12640 = Pe^{5r}$$

$$14923 = Pe^{11r}$$

As with power functions, we can solve these by dividing the second equation by the first to get

$$\frac{14923}{12640} = \frac{Pe^{11r}}{Pe^{5r}}$$

Then $1.1806 = e^{6r}$

$$\ln 1.1806 = 6r$$

$$0.1660 = 6r$$

$$r = 0.02767$$

Then we sub this into either of the equations, e.g.

$$12640 = Pe^{5r}$$

$$12640 = Pe^{0.1384}$$

$$P = \frac{12640}{e^{0.1384}}$$

$$P = 11\ 007$$

Therefore the relation is $P = 11\ 007 \times e^{0.02767t}$

If we knew the population at $t = 0$, that would make the process easier because the population at $t = 0$ is P and we only have to solve for r. We solved problems like that in Module A5-4.

Practice

Q11 The population of Grubston has increased exponentially since they held the Regional Agricultural Expo. 6 years after the expo, it was 7292; 10 years after the expo it was 8401. Find:

(a) The relation between the population and the number of years since the expo

(b) The population 4 years after the expo

(c) The number of years after the expo that the population reaches 9 000, assuming that it keeps growing at the same exponential rate

Q12 At 2 pm, Gerty put 4 mg of bacteria on a plate. The mass grew exponentially and at 6 pm there were 14 mg.

(a) How many milligrams will there be at 9 pm?
(b) When will the mass reach 50 mg?

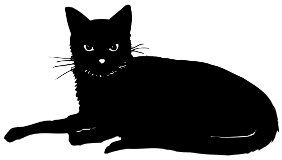
Q13 The rate at which a radioactive substance decays is proportional to the amount present. Thus the amount remaining decays exponentially. When $t = 0$, there is 24 mg; when $t = 20$, there is 17 mg.

(a) Find the relation between the amount remaining and time
(b) Find the amount present when $t = 30$
(c) Find when half the original sample has decayed.
[The time needed for half the sample to decay is called the half-life of the substance, so you calculated the half-life.]

Q14 Find the half-life of a radioactive substance if 10% of it decays in 4.23 hours.

Q15 What percentage of a radioactive substance will remain after 50 years if its half-life is 11.6 years?

A radioactive cat has 18 half-lives.



Formulae for other types of functions can be found as long as there is one known value pair for each parameter in the function, though sometimes the equations can be harder to solve.

Solve

Q51 Oranges are piled in a square pyramidal stack with 1 on the top, 4 in the next layer, 9 in the next layer, 16 in the next layer and so on. How many oranges in a stack with n layers?

Q52 Use your formula from Q51 to work out how many squares (of all sizes) on a chess board (8 by 8 small squares).

Revise

Revision Set 1

Q61 Find the formula for a linear function that passes through (–2, 7) and (3, –½).

Q62 Find the formula for the nth term of this quadratic sequence: –3, –3, 1, 9, 21.

Q63 The strength, B, of the magnetic field r cm from the end of a bar magnet along the line of the axis of the bar is given by a power function. Two measurements are shown below.

r	B
5	114.0
12	8.25

(a) Find the relation between B and r.
(b) Find the field strength when $r = 20$ cm.
(c) Find the value of r where the field strength is 1.6

Q64 Carbon-14 has a half-life of 5570 years. When a mollusc died, its shell contained 0.151 g of ^{14}C. How long ago did it die if it now contains 0.137 g?

Answers

Q1 (a) $y = 2x + 2$ (b) $y = -3x - 9$ (c) $y = \frac{1}{2}x + 5\frac{1}{2}$
Q2 (a) *fare* = *distance* × 3 + 5
Q3 (a) $y = 2x + 2$ (b) $y = -2x - 2$ (c) $y = 6x - 11$ (d) $y = -1.8x + 0.4$
Q4 (i) \$85 (ii) \$55/h (iii) \$607.50
Q5 (i) *pay* = \$8.70 + 30c/year (ii) \$47.40 per hour (iii) 36
Q6 (a) $y = x^2 + 2x - 5$ (b) $y = 2x^2 - x + 3$ (c) $y = -x^2 + 5x + 1$ (d) $y = -2x^2 + x - 6$
Q7 $h = 60 + 55t - 5t^2$
Q8 $t_n = -n^2 + 5n - 3$, where n is the term number and t_n is the nth term
Q9 (a) $p = 1.3 \times 10^{-9}\, T^4$ (b) 7.5 W (c) 937 K
Q10 (a) $r = 0.0675T^{1.25}$, where r is the rate of cooling and T is the temperature difference.
(b) 0.634°/min (c) 20.8°
Q11 (a) $p = 5896e^{0.0354t}$ (b) 6793 (c) 11.95 years
Q12 (a) 35.8 mg (b) 10:04 pm
Q13 (a) $A = 24e^{-0.01724t}$ (b) 14.3 mg (c) 40.3 years
Q14 27.8 hours
Q15 5.04%

Q51 $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$
Q52 204

Q61 $y = -1\frac{1}{2}x + 4$
Q62 $t_n = 2n^2 - 6n + 1$
Q63 (a) $B = 14\,250r^{-3}$ (b) 1.78 (c) 20.7 cm
Q64 781 years