M1 Maths

A5-5 Simultaneous Equations – General

- solve pairs of general simultaneous equations by equating, graphing and substitution
- solve larger sets of simultaneous equations

Summary Learn Solve Revise Answers

Summary

We can use graphing, equating and substitution to solve pairs of general simultaneous equations, i.e. ones that aren't necessarily linear, for example $a = b^2 + 5$, $2a + 4b = 20$.

The methods are fairly much the same as for linear equations except that solution of a non-linear equation (e.g. quadratic, exponential etc.) will be required along the way.

Larger sets of equations can also be solved, though there must be the same number of equations as there are unknowns. If all the equations are linear, then any of the four methods can be used; if any are non-linear, then we can only use graphing, equating and substitution.

Learn

The equations we solved in Module A4-3 were all linear (apart from a couple in the Solve section). They were linear in that each equation was a linear relation between the two unknowns that could be rearranged into the form $y = mx + c$.

Not all equations are linear, however. For example, in the pair $a = b^2 + 5$, $2a + 4b = 20$, the first equation is not linear.

If one or both of the equations in a pair is non-linear, we can still solve them using graphing, equating or substitution, though we can't use elimination.

Graphing

Consider the non-linear equations $h = t^2 - 2t$, $t = 3e^{h-5}$

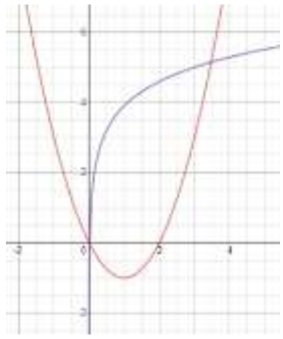
To solve these by graphing, we make the same unknown the subject of both equations. h is probably the easier choice.

$$h = t^2 - 2t$$

We rearrange the second equation like this:

$$t = 3e^{h-5}$$

$^{t}/_{3} = e^{h-5}$
$\ln {}^{t}/_{3} = h - 5$
$h = \ln {}^{t}/_{3} + 5$



Then we graph Y1 = $X^2 - 2X$ and Y2 = ln (X ÷ 3) + 5
and find intersections at (0.02, 0) and (3.48, 5.148)

So the solutions are $t = 0.02$, $h = 0$ and $t = 3.48$, $h = 5.148$

Practice

Q1 Solve each of the following pairs of equations by graphing.

(a) $2x - 3y = 5$ $\qquad y = x^2 - 2x - 1$

(b) $y = 2^x$ $\qquad 3 = 2x^2 + 5x - y$

(c) $2y = x$ $\qquad xy = 4$

(d) $m = a^3$ $\qquad a + 2m = 12$

(e) $h = 20t - 5t^2$ $\qquad h = 15 - 0.5t$

(f) $b + 5a = a^3$ $\qquad 2b + a^2 = 4$

Substituting and Equating

Suppose we have: $m - 2p = 9$, $m - p^2 = 1$

We need to rearrange one of the equations and substitute it into the other, e.g. sub $m = 9 + 2p$ into $m - p^2 = 1$ to get $9 + 2p - p^2 = 1$.

Or we can make m the subject of both equations and equate to get $9 + 2p = 1 + p^2$.

Either equation can then be rearranged into standard quadratic form $p^2 - 2p - 8 = 0$ and solved to get

$p = 4$ or $p = -2$

Subbing into the first equation, we get:

if $p = 4$, $\quad m = 9 + 2 \times 4 = 17$
if $p = -2$, $m = 9 + 2 \times -2 = 5$

So the solutions are $p = 4$, $m = 17$ and $p = -2$, $m = 5$

Subbing these results into the second equation as a check gives

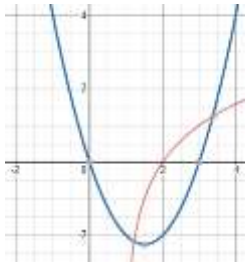
$14 = 1 + 4^2$ and $5 = 1 + (-2)^2$ which is correct.

Sometimes we produce an equation that we can't solve algebraically. For instance, if we had

$a = 2^n + 1$ and $a^2 - 3a = n$

we would generate the equation $\log_2 (a-1) = a^2 + 3a$.

This equation can only be solved by graphing.



Letting X be a and Y be n, we would graph Y1 = log (X – 1) ÷ log 2, Y2 = X^2 – 3X and find that the coordinates of the intersections are (1.222, –2.173) and (3.369, 1.244) and thus that the solutions are:

$a = 1.222$, $n = -2.173$ or $a = 3.369$, $n = 1.244$

Practice

Q2 Solve each of the following pairs of equations by equating or substitution.

(a)	$4x + 2y = 22$	$y = x^2 + x + 1$
(b)	$xy = 4$	$y = 2x + 3$
(c)	$a^2 + b^2 = 25$	$a - b = -1$
(d)	$h = 3^t$	$h + t = 4$
(e)	$s = r^3$	$s + r = 30$
(f)	$^3/_c = h + 1$	$c = h - 1$

Solving more than two simultaneous equations

We can solve for more than two unknowns, but we need one equation for each unknown. So we could solve:

$$2x + 2y - z = -11$$
$$5x - 4y + z = 26$$
$$4x - 2y + 5z = 37$$

Sets of more than two simultaneous equations cannot be solved by graphing (because we would need a graph in 3 or more dimensions) and cannot be solved by equating (because we would have two = signs in our equation). So we use elimination or substitution.

In general, if all the equations are linear, elimination is the way to go. If they are not all linear, then we have to use substitution. However, solving three equations that aren't all linear can be very messy and you will probably never need to do it, so the method isn't included here.

Linear Sets

To solve the set above, we first take one pair of the equations and eliminate one unknown from them. Then we take a different pair (though one equation will have be common to both pairs) and eliminate the same unknown from them. Then we solve those two to find the two remaining unknowns. Finally, we sub the values into any of the original equations to get the other unknown.

For the equations above, the working might look like this.

$$2x + 2y - z = -11 \text{ . . . . Eqn 1}$$

$$5x - 4y + z = 26 \text{ . . . . Eqn 2}$$

$$4x - 2y + 5z = 37 \text{ . . . . Eqn 3}$$

2 × Eqn 1 + Eqn 2 → $9x - z = 4$ . . . . Eqn 4

2 × Eqn 3 – Eqn 2 → $3x + 9z = 48$ . . . . Eqn 5

3 × Eqn 5 – Eqn 4 → $28z = 140$ . . . . Eqn 6

$$z = 5$$

Sub into Eqn 4 → $9x - 5 = 4$

$$x = 1$$

Sub these into Eqn 1 → $2 + 2y - 5 = -11$

$$2y = -8$$

$$y = -4$$

The solution is $x = 1$, $y = -4$, $z = 5$.

Practice

Q3 Solve each of the following sets of equations.

(a) $5a + 2b - 4c = 14$ $\quad$ $3a + 4b + c = 18$ $\quad$ $2a - b + 4c = 15$

(b) $a + 5b + c = -11$ $\quad$ $-a + 2b - 3c = -8$ $\quad$ $4a - b + 3c = 20$

(c) $3n + 4p - q = 29$ $\quad$ $5n - p + 2q = 10$ $\quad$ $3n + p - 3q = 14$

(d) $f - c + k = 7$ $\quad$ $2c - 5f - 2k = 7$ $\quad$ $2k + 4f = 16$

More than three equations

It is possible to solve any number of simultaneous equations as long as there is one for each unknown. To solve four, we take any three of them and eliminate one of the unknowns. Then we take a different three and eliminate the same unknown. Then we take a different three again and eliminate the same unknown. This gives us three equations in three unknowns. We then solve them as above and sub the values into any of the original equations to get the other unknown.

Solving more than three simultaneous equations can be very tedious though, and it can be hard to get through it without making an arithmetic error. For linear sets, there are easier ways using a graphics calculator which you will learn about later or using matrices, which you will learn about if you do Specialist Maths, so we don't really need to learn to do it by hand. Here's a set you can try if you'd like the challenge.

Practice

Q4 Solve the following equations:

$2a + 2b - 5c + d = 36$ $\quad$ $4a - 2b + c + 4d = -7$

$3a - 3b + 2c - d = -25$ $\quad$ $-a + 6b - 3c + 2d = 59$

Practice

Solve the following word problems:

Q5 The perimeter of a rectangular pond is 34.2 m. the area of the pond is 69.68 m^2. How long is its diagonal?

Q6 The graph of the function $y = x^2 + bx + c$ passes through the points (−3, −10) and (5, 38). What are the values of b and c?

Q7 The sum of Marnie's three children's ages is 29. The product of their ages is 864. The sum of the squares of their ages is 289. How old are they?

Solve

Q51 Harry is standing due east of Betty. Herb is standing due north of her. Herb is 217 m from Harry on a bearing of 312°. How far is Harry from Betty?

Q52 The population of Grimsberg in the second half of the 20^{th} Century could be modelled by the function $P = Ae^{kt}$, where A and k are constants and t is the number of years since 1950. If the population was 13 212 in 1952 and 10 116 in 1984, find the values of A and k and hence find the population in 1999.

Revise

Revision Set 1

Q61 Solve by graphing: $x^4 = y + 1$, $y - x = 3$

Q62 Solve by equating or substitution: $t = s^2 + 2s - 2$, $4 - t = 2s^2$

Q63 Solve the following: $a + 5b + c = -13$ $\quad -a + 2b - 3c = -25$ $\quad 4a - b + 3c = 27$

Revision Set 2

Q71 Solve by graphing: $2^x + 3 = a$, $xa = 3$

Q72 Solve by equating or substitution: $p - 6 = n^2$, $n^3 + 5n = p + n^3$

Q73 Solve the following: $x + 3y + z = -6$ $-x + 2y - 3z = -24$ $4x - y + 3z = 23$

Revision Set 3

Q81 Solve by graphing: $t = s^s$, $t + 1 = 4e^{-x}$

Q82 Solve by equating or substitution: $k = u^{2x}$, $k - 8^x = 0$

Q83 Solve the following: $2a + 5b + c = -7$ $-a + 2b - 3c = -21$ $4a - b + 3c = 39$

Answers

Q1 (a) $x = 0.279, y = -1.481$ or $x = 2.387, y = -0.075$
(b) $x = -3.018, y = 0.123$ or $x = 0.722, y = 1.649$
(c) $x = -2.828, y = -1.414$ or $x = 2.828, y = 1.414$
(d) $a = 1.725, m = 5.137$
(e) $t = 0.953, h = 14.523$ or $t = 3.147, h = 13.427$
(f) $a = -2.297, b = -0.639$ or $a = -0.397, b = 1.921$ or $a = 2.194, b = -0.407$

Q2 (a) $x = -2.952, y = 6.761$ or $x = 1.152, y = 3.479$
(b) $x = -2.351, y = -1.702$ or $x = 0.851, y = 4.702$
(c) $a = -4, b = -3$ or $a = 3, b = 4$
(d) $h = 1, t = 3$
(e) $r = 3, s = 27$
(f) $c = -3, h = -2$ or $c = 1, h = 2$

Q3 (a) $a = 4, b = 1, c = 2$ (b) $a = 5, b = -3, c = -1$
(c) $n = 3, p = 5, q = 0$ (d) $f = 3, c = -2, k = 2$

Q4 $a = 2, b = 8, c = -3, d = 1$

Q5 12.37 m

Q6 $b = 4, c = -7$

Q7 12, 9, 8

Q51 161 m Q52 $A = 13434$, $k = -0.008344$, population = 8926

Q61 $x = -1.284, y = 1.716$ or $x = 1.534, y = 4.534$

Q62 $s = -1.786, t = -2.382$ or $s = 1.12, t = 1.493$

Q63 $a = 2, b = -4, c = 5$

Q71 $x = 0.656, a = 4.575$

Q72 $n = 2, p = 10$ or $x = 3, y = 15$

Q73 $x = 1, y = -4, z = 5$

Q81 $s = 0.784, t = 0.826$

Q82 $u = 0, k = 1$

Q83 $a = 2, b = -4, c = 9$