M1 Maths

A5-4 Exponential Functions and Logs

- general form and graph shape
- equation solution methods and logs
- applications

Summary   Learn   Solve   Revise   Answers

Summary

The general form of exponential functions is $y = Pa^x$, where P and a are the parameters and a is positive. Graphs of exponential functions pass through the point $(0, P)$; if $a > 1$; the gradient is positive and increases as x increases, if $a < 1$, the gradient is negative and becomes less negative as x increases.

Exponential functions are used in situations where the dependent variable is multiplied by the same factor for each increment in the independent variable.

Exponential equations are solved by taking logs of both sides.

The number e is often used as the base in exponential functions. The general form is then $y = Pe^{rx}$.

Learn

General Form and Shape of the Graph

You should already know how to calculate compound interest (Module N4-1). The relation between the amount of money and time is $A = P \times (1 + r)^t$, where A is the amount, P is the principal, r is the interest rate as a decimal and t is time.

If $P = 500$ and $r = 0.06$, then $A = 500 \times 1.06^t$.

This is an exponential relation: A is an exponential function of t. Exponential functions are so called because the independent variable is the exponent. *Exponent* is another word for *index*.

The general form of an exponential function is $y = Pa^x$.

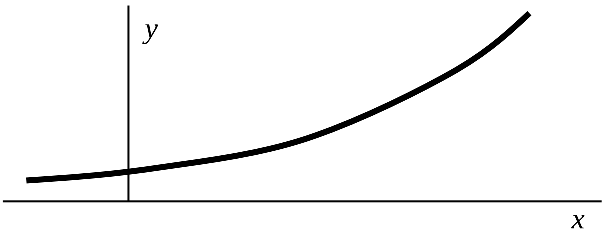
y is the dependent variable. In many cases we use A rather than y because it is often an amount of something. x is the independent variable, though the independent variable is often time and then we tend to call it t. So $A = Pa^t$ is commonly used.

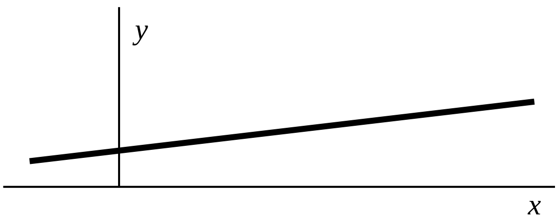
The parameter P is the initial amount, i.e. the amount when $x = 0$. It is sometimes called the principal (hence the letter P) and is sometimes written as A_0 (A when $t = 0$). a is the growth factor, i.e. what the amount gets multiplied by each time x increases by 1 unit. a must be positive to avoid fractional powers of negative numbers.

Occurrence

Exponential functions are used to describe situations where the dependent quantity changes by being multiplied by the same amount with each unit increase in the independent quantity. This is the same as saying that the dependent quantity increases by the same fraction or percentage with each unit increase in the independent quantity. For example, in the case of compound interest, the dependent variable (amount of money) might increase by 6% for each unit increase in time (i.e. each year).

Because the amount gets progressively bigger, the increase also gets progressively bigger, so the amount increases at an increasing rate. The gradient of the graph therefore gets progressively steeper as in the graph on the left below.



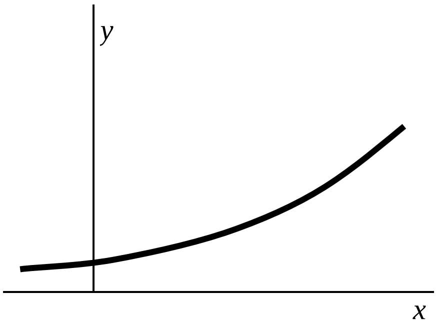


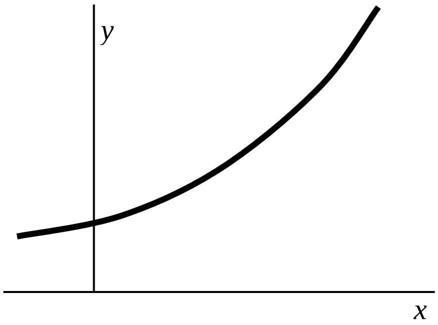
This type of growth is called exponential growth. Another common type of growth is linear growth. With linear growth, the dependent variable increases by the same amount (rather than the same percentage) with each unit increase in the independent variable. Linear growth is represented by a linear function and looks like the graph on the right above.

Effect of the parameters on the shape of the graph

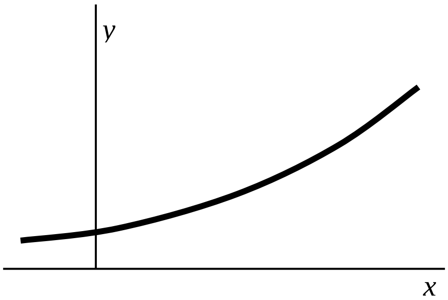
The general form of an exponential function is $y = Pa^x$. P is the initial amount, the amount when $x = 0$. This is the y-value where the graph crosses the y-axis, i.e. it is the y-intercept. a is the growth factor. The higher a is, the faster the graph rises as x increases.

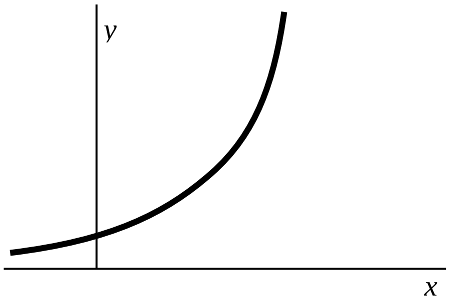
Here are two exponential functions. They have the same value for a, but the one on the right has the higher value for P.





Here are two more exponential functions. They have the same value for P, but the one on the right has the higher value for a.

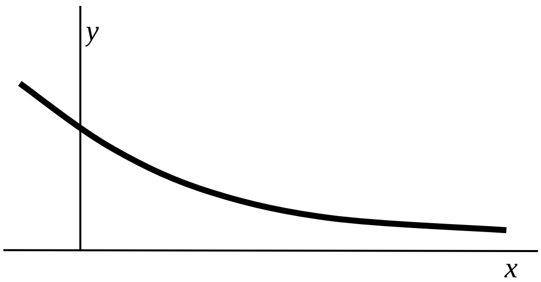




If the amount increases by 6% each unit increase in x, then the growth factor, a, will be 106% or 1.06. If the amount decreases by 6% each year, then the growth factor, a, would then be 94% or 0.94.

The equation would then be $y = P \times 0.94^x$.

Over time, the amount of money would decrease, as shown in the next graph.



This is called exponential decay. Exponential decay happens when the growth factor is less than 1.

Practice

Q1 Without a graphics calculator, sketch the following exponential functions:

(a) $y = 3 \times 1.2^x$ (b) $y = 2^x$

(c) $h = ½ \times 4^t$ (d) $v = 0.8^x$

(e) $a = 36 \times 4^x$ (f) $y = 20 \times 0.2^x$

Use your graphics calculator to check your sketches.

Solving Equations from Exponential Functions

When we solve linear equations, we proceed by using inverse operations to undo the things that were done to the unknown, starting with the last one. For instance, to solve $2x + 4 = 14$, we would subtract 4, then divide by 2.

The operation that changes x to 10^x is called exponentiating or exponentiation. The inverse operation of exponentiating is called taking a log.

We solve the exponential equation $10^x = 50$, by taking logs of both sides.

Taking a log of 10^x just undoes the exponentiation and takes us back to x. When we take a log of 50, the result is written as log 50. So we get $x = \log 50$.

There is a log button on your calculator. You just press log, then press 50 and you should get 1.699. This is log 50 and the solution to the equation, i.e. the value of x.

The complete solution will look like this:

$$10^x = 50$$
$$x = \log 50$$
$$x = 1.699$$

Practice

Q2 Solve the following equations:

(a) $10^x = 70$ (b) $10^x = 4500$ (c) $10^x = 5$ (d) $10^x = 0.4$

(e) $10^t = 12.6$ (f) $10^t = 0.0034$ (g) $10^t = 10$ (h) $10^t = 1$

We can think of log 50 as the power to which we have to raise 10 to get 50. $10^{1.669} = 50$. In the same way $10^2 = 100$, so $\log 100 = 2$, $10^3 = 1000$, so $\log 1000 = 3$, and so on.

Other Bases

These logs are called logs base 10 and are suitable for solving equations of the form $10^x = 42$. But what if we wanted to solve $5^x = 42$. Here the exponentiation has a base of 5 instead of 10, so we need a log base 5, i.e. the power to which we have to raise 5 to make 42.

Because logs base 10 are very commonly used, when we write log 42, it is assumed to be log base 10 of 42. If we mean log base 5 of 42, then we write $\log_5 42$. Ditto for any other base other than 10. Of course, we can write logs base 10 as $\log_{10} 42$ as well if we like, but we don't have to.

Now your calculator may be able to do logs base 5, but it may not. If it doesn't, then the log button on the calculator will give you a log base 10.

There is a way of getting a log base 5 though. We use the 'change of base rule' which says that $\log_5 42$ is equal to $\frac{\log_{10} 42}{\log_{10} 5}$, i.e. $\frac{\log 42}{\log 5}$. So we enter log 42 ÷ log 5 and it will give us 2.32. This is the power to which we have to raise 5 to make 42: $5^{2.32} = 42$.

Similarly for any other base: $\log_2 2.9 = \frac{\log 2.9}{\log 2}$, $\log_{12.6} 0.034 = \frac{\log 0.034}{\log 12.6}$, and in general $\log_b a = \frac{\log a}{\log b}$.

One way to remember which way round log a and log b go is to remember that the number which is higher up the page in the $\log_b a$ is also higher up the page in the $\frac{\log a}{\log b}$.

Practice

Q3 Solve the following equations using logs:

(a) $5^t = 30$	(b) $2^t = 9.4$	(c) $20^t = 214$	(d) $1.06^t = 2$
(e) $8^t = 4000$	(f) $10^t = 0.0052$	(g) $0.5^t = 0.0016$	(h) $5^t = 0.019$

Multi-step Equations

There may be other steps to undo in exponential equations. Then we just undo each operation in turn, starting with the last one.

For instance, if we needed to solve $4 \times 3.5^{2x-5} + 11 = 28$, we would proceed like this:

$$4 \times 3.5^{2x-5} + 11 = 28$$
$$4 \times 3.5^{2x-5} = 17$$
$$3.5^{2x-5} = 4.25$$

$$2x - 5 = \log_{3.5} 4.25$$

$$2x - 5 = \frac{\log 4.25}{\log 3.5}$$

$$2x - 5 = 1.155$$

$$2x = 6.155$$

$$x = 3.077$$

Practice

Q4 Solve the following equations using logs:

(e) $5 \times 2^t = 60$ (f) $0.4 \times 2^t = 6.88$ (g) $1.6 \times 12^t = 21$ (h) $4 \times 1.16^t = 1.8$

(i) $5.5^{4t} = 42$ (j) $10^{0.01t} = 279$ (k) $5^{2t+1} \times 6 = 214$ (l) $2 \times 6^{0.5t-4} + 5 = 9$

(m) $\frac{4 \times 10^{-0.004t} + 15}{7} = 30$ (n) $2 \times 2^{-5t} = 0.004$

Note: The word 'log' is actually short for 'logarithm', though the word 'logarithm is rarely used. It is nothing to do with cylindrical hunks of wood.

Other applications of exponential functions

Exponential functions can be used to describe anything which grows or decays exponentially i.e. is multiplied by the same number in each time interval.

The number of bacteria growing in a culture increases exponentially as long as there is enough food and room. With radioactive decay, the amount of the isotope remaining decreases exponentially. The value of an item that depreciates at a certain percentage per year decays exponentially. And so on.

Suppose that a bacterial colony weighs 5 mg when $t = 0$ and then doubles in mass every hour. The mass, m, present at any time t is given by

$m = 5 \times 2^t$ where m is in milligrams and t is in hours

If we want to know when the mass will reach 1000 mg, we sub 1000 for m to get $1000 = 5 \times 2^t$. Then we solve this to get $t = 7.64$ h.

Practice

Q5 Jeremy put $2000 in the bank and got 1% per month compound interest.

(a) How much would he have after 72 months?

(b) How many months would he have to wait for his deposit to grow to

$10 000?

Q6 A bacterial colony doubles its mass every hour. To the nearest minute, how long will it take to grow from 10 mg to 200 mg?

Q7 Another bacterial colony triples its mass every hour. To the nearest minute, how long will it take to grow from 1 mg to 200 mg?

Q8 Harriet's car loses 10% of its value every year. How long will it take to drop in value by 90%?

Q9 A pool filter cleans out 40% of the dirt every hour.

(a) How long will it take to clean out 95% of the dirt? [Careful, it's more than 2½ hours.]

(b) How long will it take to clean out all of it?

Q10 The population of the world grows by 2% per year. At that rate,

(a) if the population is presently 7.5 billion, what will it be in 15 years time?

(b) how long would it take to grow from 7.5 billion to 20 billion?

Using *e* as a base

It is customary, when writing formulae for exponential functions, to use the number *e* as the base.

e is an irrational number whose approximate value is 2.718281828. Its exact value is $\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+$....... continued for ever. [3! is pronounced '*3 factorial*' and means 3×2×1; 7! means 7×6×5×4×3×2×1; 0! is taken to be 1.]

The inverse operation of exponentiation base *e* is of course taking a log base *e*. So $\log_e e^x = x$. $\log_e$ is often shortened to ln, which stands to 'log natural' (natural log) and which is pronounced 'lin' as in the girl's name. So $\log_e x = \ln x$, which is pronounced 'lin *x*'. You will notice the ln button next to the log button on your calculator. This allows us to get logs base *e* and to simply solve exponential equations base *e*.

e is used as a base partly because it makes the calculus of exponential functions simpler, as you will see if you do calculus later.

When using base *e*, different bases can be reproduced by introducing the constant *r* to get $e^{rx.}$ It is quite easy to change other bases to base *e* and vice versa. Suppose we have 2^x and we wish to change the base from 2 to *e*. We write 2 as $e^{\ln 2}$. Then 2^x becomes $(e^{\ln 2})^x$, which is $e^{\ln 2 \times x}$. In this case, the constant *r* is ln 2 which is about 0.693. So 2^x is written as $e^{0.693x}$.

In general, if $y = a^x$, then $y = e^{rx}$ where $r = \ln a$.

Changing the other way is just the reverse process. If we have e^{-5t}, we call it $(e^{-5})^t$. The base is then e^{-5}, which is 0.006738. So $e^{-5t} = 0.006738^t$.

In general, if $y = e^{rx}$, then $y = a^x$, where $a = e^r$.

To solve an equation like $e^{2x} = 24$, however, the easiest way is to take logs base e, then divide by 2.

$$e^{2x} = 24$$
$$2x = \ln 24$$
$$= 3.178$$
$$x = 1.589$$

Practice

Q11 Write the following expressions with base e.

(a) 2^x (b) 10^x (c) 0.4^x

(d) 1.08^x (e) 0.031^x (f) 100^x

Q12 Write the following expressions in the form a^x.

(a) e^x (b) e^{3x} (c) $e^{0.5x}$

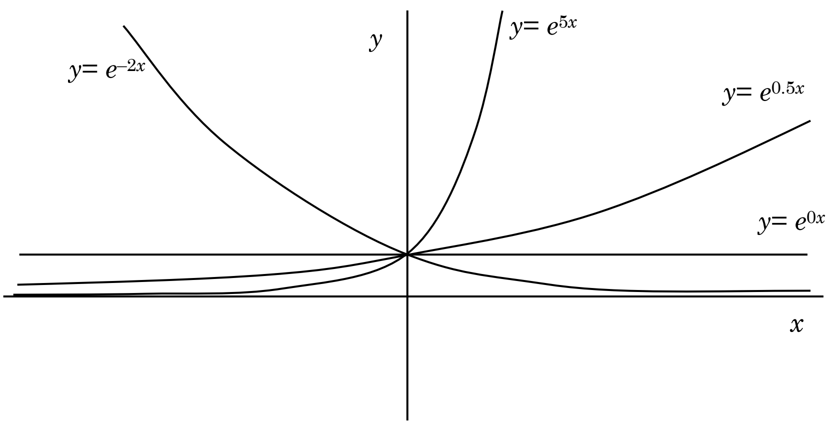
(d) e^{-2x} (e) $e^{-0.2x}$ (f) $e^{0.003x}$

Q13 Use the ln button on your calculator to solve:

(a) $e^x = 38$ (b) $e^x = 1.44$ (c) $e^x = 0.02$

(d) $e^{2x} = 24$ (e) $e^{-0.04x} = 1.09$ (f) $e^{0.6x} = 0.62$

The graph of $y = e^{rx}$ rises exponentially if $r > 0$ and decays exponentially if $r < 0$. If $r = 0$, it is horizontal.



Another advantage of using the $y = e^{rx}$ form is that r is the gradient of the graph at $x = 0$ and is the fractional instantaneous growth rate at any time. Hence the letter r.

Mathematicians see e as a number as fundamental to mathematics as π. In fact, many mathematicians consider the five most fundamental numbers to be 0, 1, e, π and i (i is $\sqrt{-1}$, an imaginary number). These five numbers are related by the identity $e^{i\pi} + 1 = 0$. This is known as Euler's identity (named after the 18th Century Swiss mathematician Leonhard Euler (pronounced 'oiler')) and it excites mathematicians as much as $E = mc^2$ excites physicists.

Problems with Base e

Suppose that the population of cane toads in a newly invaded area grows exponentially. At the start of 2020, there are 4000. At the start of 2025, there are 23 500. How many will there be at the start of 2028 and when will the population reach 100 000?

We can solved this as follows.

We know the growth is exponential, so we can write the relation between number, n, and time in years since 2020, t, as $n = Pe^{rt}$. We can then substitute for the variables whose values we know and solve equations to find the values of the ones we don't know.

In this case, we are told that, when $t = 0$, $n = 4000$,

$$\text{so } 4000 = Pe^{r\times 0}$$

$$4000 = Pe^{0}$$

$$4000 = P \times 1$$

$$4000 = P$$

$$\text{So } n = 4000e^{rt}$$

Now we are also told that, when $t = 5$, $n = 23\ 500$,

$$\text{so } 23\ 500 = 4000e^{5r}$$

$$5.875 = e^{5r}$$

$$\ln 5.875 = 5r$$

$$1.771 = 5r$$

$$0.354 = r$$

$$\text{So } n = 4000e^{0.354t}$$

This is the specific relation between n and t for the toads. We can use this to find n for a given value of t or to find t for a given value of n.

At the start of 2028, $t = 8$,

$$\text{so } n = 4000e^{0.354 \times 8}$$

$$= 67\,918.$$

So there will be 67 918 toads at the start of 2028.

(Note, $4000e^{0.354 \times 8}$ can be put into the calculator in one go using the e^x button (2nd function of ln x), but you must put the 0.354×8 in brackets.)

When the population reaches 100 000,

$$100\,000 = 4000e^{0.354t}$$

$$25 = e^{0.354t}$$

$$\ln 25 = 0.354t$$

$$3.219 = 0.354t$$

$$t = 9.093$$

So the population will reach 100 000 early in 2029.

This is quite a long solution with several steps, but it is important to be able to solve this type of problem. It is quite a fundamental and widely used procedure used in many more advanced algebra problems. Mastering it will make you a much more accomplished mathematician. Also, you will come across many problems requiring a similar approach if you go on to do calculus.

Read through the solution above until it all makes sense and the way to go at each step is obvious. Then do the following practice questions.

Practice

Use $A = Pe^{rt}$ to solve the following problems

Q14 The population of cane toads grows exponentially after the first few are released. The number, n, t years after release is given by $n = 40e^{1.2t}$. Find:

(a) How many there would be 5 years after the initial release

(b) How long it would take for the population to grow to 1 000 000.

Q15 The population of mice in a barn grows exponentially. When first noticed, there are 57. 80 days later, there are 490. Find:

(a) How many there would have been 50 days after they were first noticed

(b) When the population would reach 1000.

Q16 A radioactive isotope decays such that the amount remaining decreases exponentially. When it was first received, there was 2.70 mg. 6 days later, there was 1.94 mg. Find:

(a) How much was left 2 days after it was received

(b) How long it would take for half of it to decay.

Q17 A different radioactive isotope also decays such that the amount remaining decays exponentially. It has a half-life of 12.73 years. This means that half the initial amount remains 12.73 years later. Find:

(a) The fraction remaining after 60 years

(b) The time needed for 10% of it to decay.

Hint: you can pick any amount for the initial amount (at $t = 0$). It is easy to make it 1 gram.

Q18 A bacterial colony triples its mass every hour.

(a) To the nearest minute, how long will it take to grow from 1 mg to 200 mg?

(b) By what factor would the mass increase in 20 minutes?

Q19 Sadie's car decreased in value by 55% in 5 years. Assuming the decrease is exponential,

(a) By what percentage does it drop each year?

(b) How long will it take to be worth nothing?

Solve

Q51 Hercules puts $1000 in the bank at 5% per annum interest compounding yearly. Hieronymus put $1000 in at 7.2% simple interest. A year later, Hieronymus' account will have more money than Hercules'. But some time later, Hercules' account will overtake Hieronymus'. When will that be?

Q52 The atmosphere contains carbon 12 (mostly as CO2) and lesser amounts of carbon 13 and carbon 14. Carbon 12 and Carbon 13 are stable, but carbon 14 decays with a half-life of 5570 years. Carbon 14 is renewed by the action of cosmic rays on the atmosphere. When plants grow, they absorb carbon from the atmosphere (as CO_2). The ratio of carbon 14 to carbon 12 is about $1:1.05 \times 10^{12}$. and animals that eat those plants have about the same ratio in their bones. A human bone was found to have a ratio of $1: 1.67 \times 10^{12}$. How old would it be?

Q53 The rate of convective heat loss from a body is proportional to the temperature difference between the body and the air raised to the power of $^5/_4$. When the temperature difference is 20°, the rate of heat loss is 3.4 Joules / second. What will be the rate when the temperature difference is 12°?

Q54 A challenge: $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Prove that $e^{ix} = \cos x + i \sin x$, where $i = \sqrt{-1}$

Revise

Revision Set 1

Q61 Give the general form of exponential functions.

Q62 On the same axes, sketch the graphs of $y = 2^x$, $y = 3 \times 2^x$, $y = 5^x$, $y = 0.7^x$
Check your sketches with the graphics calculator.

Q63 Solve:
(a) $2^x = 5$ (b) $4 \times 5^x = 12.4$ (c) $200 \times 1.09^x = 6000$

Q64 The volume, V, (in litres) of water left in a leaking tank at time t (in hours) is given by $V = 3000 \times 0.912^t$. Find
(a) how much water is left when $t = 12$
(b) when the tank will contain 1500 L.

Revision Set 2

Q71 Give the general form of exponential functions.

Q72 On the same axes, sketch the graphs of $y = 3^x$, $y = 2 \times 3^x$, $y = 5^x$, $y = 0.5^x$
Check your sketches with the graphics calculator.

Q73 Solve:
(a) $1.2^x = 5$ (b) $4 \times 5^{2x} = 11$ (c) $50 \times 1.09^{(x+1)} = 77$

Q74 \$3 000 is invested at 8% p.a. compound interest. Find
(a) how much will be in the account after 5 years
(b) how long it will take for the money to double.

Revision Set 3

Q81 Give the general form of exponential functions.

Q82 On the same axes, sketch the graphs of $y = 2^x$, $y = 3 \times 2^x$, $y = 5^x$, $y = 0.7^x$
Check your sketches with the graphics calculator.

Q83 Solve:

(a) $3^x = 32$ (b) $2 \times 10^x = 38.4$ (c) $1000 \times 1.04^x = 560$

Q84 The value of a car depreciated by 10% each year. If the value when new is $28 000, find

(a) the value when it is 6 years old

(b) when it will be worth $4000.

Answers

Q2	(a) 1.85	(b) 3.65	(c) 0.699	(d) −3.98
	(e) 1.10	(f) −2.47	(g) 1	(h) 0
Q3	(a) 2.11	(b) 3.23	(c) 1.79	(d) 11.9
	(e) 3.99	(f) −2.28	(g) 9.29	(h) −2.46
Q4	(a) 3.58	(b) 4.10	(c) 1.04	(d) −5.38
	(e) 0.548	(f) 245	(g) 0.610	(h) 8.77
	(m) −421	(n) 1.79		
Q5	(a) $4094.20	(b) 162 months		
Q6	7 h 39 min	Q7. 4 h 49 min	Q8. 30 years	
Q9	(a) 5.86 h	(b) it will never clean out all the dirt		
Q10	(a) 10.1 billion	(c) 49.5 years		
Q11	(a) $e^{0.693x}$	(b) $e^{2.30x}$	(c) $e^{-0.916x}$	
	(d) $e^{0.077x}$	(e) $e^{-3.47x}$	(f) $e^{4.61x}$	
Q12	(a) 2.72^x	(b) 20.1^x	(c) 1.65^x	
	(d) 0.135^x	(e) 0.819^x	(f) 1.003^x	
Q13	(a) 3.64	(b) 0.365	(c) −3.91	
	(d) 1.59	(e) −2.15	(f) −0.797	
Q14	(a) 16 137	(b) 8.44 years		
Q15	(a) 219	(b) After 106.5 days		
Q16	(a) 2.42 mg	(b) 30.6 days		
Q17	(a) 3.8%	(b) 1.935 years		
Q18	(a) 4 h 49 min	(b) 1.44		
Q19	(a) 14.8%	(b) It will never be worth nothing.		
Q51.	15.04 years after making the deposits		Q52 3729 years	Q53 1.80 J/s
Q61	$y = pa^x$			
Q63	(a) 2.32	(b) 0.703	(c) 39.5	
Q64	(a) 993 L	(b) $t = 7.52$		
Q71	$y = pa^x$			
Q73	(a) 8.83	(b) 0.314	(c) 4.01	
Q74	(a) $4407.98	(b) 9.01 years		
Q81	$y = pa^x$			
Q83	(a) 3.15	(b) 1.28	(c) −14.8	
Q84	(a) $14 880	(b) After 18.47 years		