M1 Maths

Learning by Thinking

A5-2 Index Laws 6-10

- zero, negative and fractional indices
- converting between fractional indices and surds

Learn Answers

This LbT (Learning by Thinking) module is an alternative to the ‘Learn’ section of the normal module. It is designed to lead the student to work out the maths themselves by solving problems. Thus it contains only minimal explanations. The rationale behind the approach can be read **here**.

Learn

Definitions of a^n when n is not a natural number

In Module A3-10 we defined a^n as n lots of a multiplied together.

So $a^2 = a\times a$, $a^5 = a\times a\times a\times a\times a$, $a^1 = a$ etc.

Actually, we didn’t talk much about a^1, because we generally just call it a, but, because it isn’t as obvious as higher powers, the fact that $a^1 = a$ is often given as a sixth index law.

This definition of a^n can’t be applied to zero, negative or fractional values of n, however, because you can’t multiply no as together or half an a together or –2 as together. We could define a^n where n is not a natural number in any way we like. But, of course, life is easier if we define it in such a way that Index Laws 1-5 that we met in A3-10 are still true when n is not a natural number.

If we define them as in Laws 6 to 10 below, then Laws 1 to 5 will still hold. So this is what we do.

Law 6: a^1

As explained above $a^1 = a$.

Q1 Evaluate:

(a) 2^1 (b) 16^1 (c) 4.3^1 (d) 1^1 (e) 0^1

Q2 Simplify:

(a) a^1 (b) $a^2 \times a$ (c) $(a^5)^1$ (d) $x \div x^3$ (e) $(-4)^1$

Law 7: a^0

We want to define a^0 such that the index laws still hold.

When we define it, the following must be true: $a^n \times a^0 = a^{n+0} = a^n$

Q3 For that to be true, what must a^0 be?

Q4 Evaluate without a calculator:

(a) 2^0 (b) 72.9^0 (c) p^0 (d) 1^0 (e) $(2x^2z)^0$ (f) 0^0

Note on (f): By our definition $0^0 = 1$. But 0 to the power of anything equals 0. To avoid this inconsistency, we leave 0^0 out of the definition and say $a^0 = 1$ if $a \neq 0$. 0^0 is undefined in the same way that $0 \div 0$ is undefined.

So Law 7 says that $a^0 = 1$ as long as $a \neq 0$. In other words, anything to the power of zero is 1 except 0^0 which is not defined.

If we define it that way, the other index laws will hold. Check them if you wish.

Law 8: a^{-n}

We want to define a^{-n} such that the index laws still hold.

When we define it, the following must be true: $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$

Q5 For that to be true, what must a^{-n} be?

Q6 Evaluate without a calculator:

(a) 2^{-2}	(b) 10^{-3}	(c) 7^{-1}	(d) 1^{-8}
(e) 2^{-4}	(f) 0.5^{-2}	(g) 5^{-1}	(h) $(½)^{-3}$
(i) 0.25^{-2}	(j) $(^3/_2)^{-1}$	(k) $(^4/_5)^{-2}$	(l) $(^5/_2)^{-3}$

Q7 Write as fractions without negative indices:

(a) a^{-2}	(b) s^{-3}	(c) t^{-1}	(d) k^{-8}
(e) $2w^{-3}$	(f) $5v^{-1}$	(g) $(3p)^{-4}$	(h) $3(2ar)^{-5}$
(i) a^3x^{-3}	(j) $2s^{-5}t^2$	(k) $2u^{-1}v^4w^{-3}$	(l) $½g^2h^{-5}p^{-3}$

Q8 Write these using negative indices rather than fractions:

(a) $\frac{1}{b^2}$	(b) $\frac{1}{e^3}$	(c) $\frac{1}{r^{10}}$	(d) $\frac{1}{a}$
(e) $\frac{2}{w^2}$	(f) $\frac{10}{h^5}$	(g) $\frac{1}{2b^2}$	(h) $\frac{3}{a^3}$
(i) $\frac{a^3}{c^2}$	(j) $\frac{4a^3}{tc^2}$	(k) $\frac{x^4}{4a^3r^2}$	(l) $4 \times \frac{3s^3}{6c^2}$

Q9 Simplify, giving the answers without negative indices:

(a) $a^6 \times a^{-2}$ (b) $c^{-4} \times c^3$ (c) $h^{-3} \times h^{-1}$ (d) $(r^{-3})^2$

(e) $t^{-3} \div t^5$ (f) $k^{-2} \div k^{-5} \times d^{-2}$ (g) $s^{-1} \times r^{-3} \times s^4$ (h) $(p^2 \div p^{-4})^{-3} \div p^3$

(i) $\frac{a^2v^{-3}x^{-4}}{4(v^3x^{-1})^{-2}} \div \frac{a^{-3}v^2x}{(2v^2x^{-1})^{-3}a}$

So Law 8 says: $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$ as long as $a \neq 0$.

In other words a^{-n} is the reciprocal of a^n (as long as a is not 0).

Note: Law 8 applies only if $a \neq 0$ because, by this definition $0^{-n} = \frac{1}{0^n}. = \frac{1}{0}$. But division by 0 is not defined.

Law 9: $a^{1/n}$

We want to define a^0 such that the index laws still hold.

When we define it, the following must be true: $(a^{1/n})^n = a^{1/n \times n} = a^1 = a$

Q10 For that to be true, what must $(a^{1/n})$ be?

Q11 Evaluate without a calculator:

(a) $4^{1/2}$ (b) $8^{1/3}$ (c) $16^{1/2}$ (d) $16^{0.5}$

(e) $32^{0.2}$ (f) $9^{-1/2}$ (g) $36^{-0.5}$ (h) $16^{-0.25}$

Q12 Write as single powers:

(a) $\sqrt{6}$ (b) $\sqrt[3]{10}$ (c) $\sqrt[5]{x}$ (d) $\sqrt[4]{xy}$

Q13 Write in surd form (i.e. with roots):

(a) $5^{1/5}$ (b) $2^{-1/2}$ (c) $a^{0.25}$ (d) $7^{-0.2}$

(e) $\frac{1}{2^{-0.1}}$ (f) $w^{-1/3}$ (g) $x^{0.5}$ (h) $d^{-0.25}$

So Law 9 says $a^{1/n} = \sqrt[n]{a}$, the n^{th} root of a, i.e. the number which, when raised to the power of n, makes a.

Law 10: $a^{m/n}$

Q14 Use Laws 1-9 to show that $\boldsymbol{a^{m/n} = (\sqrt[n]{a})^m}$

Q15 In the same way show that $\boldsymbol{a^{m/n} = \sqrt[n]{a^m}}$

Q16 Evaluate without a calculator:

(a) $8^{2/3}$ (b) $9^{3/2}$ (c) $25^{1.5}$ (d) $32^{1.2}$

(e) $27^{-4/3}$ (f) $32^{-0.6}$ (g) $1024^{-0.3}$ (h) $4^{4.5}$

(i) $(-8)^{2/3}$ (j) $(-8)^{5/3}$ (k) $(-8)^{-2/3}$ (l) $9^{-1.5}$

Q17 Simplify:

(a) $u^{1/2} \times u^{5/2}$ (b) $u^{1/2} \times u^{1/3}$ (c) $h^{2/3} \div h^{-1/6}$

Q18 Write as single powers:

(a) $\sqrt{5^3}$ (b) $\left(\sqrt{5}\right)^3$ (c) $2\sqrt{2}$ (d) $a\left(\sqrt[5]{a}\right)$ (e) $s \div \sqrt[3]{s}$ (f) $\sqrt{c} \div \sqrt[6]{c}$

Q19 Write in surd form (i.e. with roots):

(a) $3^{2/5}$ (b) $s^{13/4}$ (c) $w^{-2/3}$ (d) $x^{0.4}$ (e) $d^{-2.1}$ (f) $\frac{1}{2^{-0.6}}$

Note on irrational indices

We now have definitions for powers with any rational index. But we have not defined powers with irrational indices. It turns out that there is no real need to do that. So we don't.

$2^{\sqrt{2}}$ is undefined, as is 2^{π}, though, of course $2^{3.1415926}$ is defined. It is approximately 8.825.

Q20 $(-8)^{-5/2}$ isn't a real number. Explain why. What other similar expressions aren't real for the same reason?

Summary of all 10 Index Laws

Below is a summary of the 10 index laws

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $a^m \div a^n = a^{m-n}$

Law 3: $(a^m)^n = a^{mn}$

Law 4: $(ab)^n = a^n b^n$

Law 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Law 6: $a^1 = a$

Law 7: $a^0 = 1$

Law 8: $a^{-n} = \frac{1}{a^n}$

Law 9: $a^{1/n} = \sqrt[n]{a}$

Law 10: $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Answers

Q1	(a) 2	(b) 16	(c) 4.3	(d) 1	(e) 0	
Q2	(a) a	(b) a^3	(c) a^5	(d) $1/x^3$	(e) -4	
Q3	1					
Q4	(a) 1	(b) 1	(c) 1	(d) 1	(e) 1	(f) not defined
Q5	$\frac{1}{a^n}$					
Q6	(a) ¼	(b) 0.001	(c) $^1/_7$	(d) 1		
	(e) $^1/_{16}$	(f) 4	(g) $^1/_5$	(h) 8		
	(i) 16	(j) $^2/_3$	(k) $^{25}/_{16}$	(l) $^8/_{125}$		
Q7	(a) $\frac{1}{a^2}$	(b) $\frac{1}{s^3}$	(c) $\frac{1}{t}$	(d) $\frac{1}{k^8}$		
	(e) $\frac{2}{w^3}$	(f) $\frac{5}{v}$	(g) $\frac{1}{81p^4}$	(h) $\frac{3}{32a^5r^5}$		
	(i) $\frac{a^3}{x^3}$	(j) $\frac{2t^2}{s^5}$	(k) $\frac{2v^2}{uw^3}$	(l) $\frac{g^2}{2h^5p^3}$		
Q8	(a) b^{-2}	(b) e^{-3}	(c) r^{-10}	(d) a^{-1}		
	(e) $2w^{-2}$	(f) $10h^{-5}$	(g) $\frac{1}{2}b^{-2}$	(h) $3a^{-3}$		
	(i) a^3c^{-2}	(j) $4a^3t^{-1}c^{-2}$	(k) $\frac{1}{4}x^4a^{-3}r^{-2}$	(l) $2s^3c^{-2}$		
Q9	(a) a^4	(b) $\frac{1}{c}$	(c) $\frac{1}{h^4}$	(d) $\frac{1}{r^6}$		
	(e) $\frac{1}{t^8}$	(f) $\frac{k^3}{d^2}$	(g) $\frac{s^3}{r^3}$	(h) $\frac{1}{p^{21}}$		
	(i) $\frac{a^6}{32v^5x^4}$					
Q10	$\sqrt[n]{a}$					
Q11	(a) 2	(b) 2	(c) 4	(d) 4		
	(e) 2	(f) $^1/_3$	(g) $^1/_6$	(h) ½		
Q12	(a) $6^{1/2}$	(b) $10^{1/3}$	(c) $x^{1/5}$	(d) $(xy)^{1/4}$		
Q13	(a) $\sqrt[5]{5}$	(b) $1/\sqrt{2}$	(c) $\sqrt[4]{a}$	(d) $1/\sqrt[5]{7}$		
	(e) $\sqrt[10]{2}$	(f) $1/\sqrt[3]{w}$	(g) $\sqrt{x}$	(h) $1/\sqrt[4]{d}$		
Q14	$\boldsymbol{a^{m/n} = a^{1/n \times m} = (a^{1/n})^m = (\sqrt[n]{a})^m}$					
Q15	$\boldsymbol{a^{m/n} = a^{m \times 1/n} = (a^m)^{1/n} = \sqrt[n]{a^m}}$					
Q16	(a) 4	(b) 27	(c) 125	(d) 64		
	(e) $^1/_{81}$	(f) $^1/_8$	(g) $^1/_8$	(h) 512		
	(i) 4	(j) -32	(k) ¼	(l) $^1/_{27}$		
Q17	(a) u^3	(b) $u^{5/6}$	(c) $u^{5/6}$			
Q18	(a) $5^{3/2}$	(b) $5^{3/2}$	(c) $2^{3/2}$	(d) $a^{6/5}$	(e) $s^{2/3}$	(f) $c^{5/6}$
Q19	(a) $\sqrt[5]{3^2}$	(b) $\sqrt[4]{s^7}$	(c) $1/\sqrt[3]{w^2}$	(d) $\sqrt[5]{x^2}$	(e) $1/\sqrt[10]{d^{21}}$	(f) $\sqrt[5]{2^3}$

Q20 It involves the square root of a negative number. Any negative number raised to a fractional power where the index in simplest form has an even denominator will not be real.