M1 Maths

A5-2 Index Laws 6-10

- zero, negative and fractional indices
- converting between fractional indices and surds

Summary Learn Solve Revise Answers

Summary

Index laws 6-10 define the meaning of powers with zero, negative and fractional indices. For any real number a (except 0 in the case of Laws 7 and 8) and any counting number n,

Law 6: $a^1 = a$

Law 7: $a^0 = 1$

Law 8: $a^{-n} = \frac{1}{a^n}$

Law 9: $a^{1/n} = \sqrt[n]{a}$

Law 10: $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Learn

Definitions of a^n when n is not a natural number

In Module A3-10 we defined a^n as n lots of a multiplied together.

So $a^2 = a \times a$, $a^5 = a \times a \times a \times a \times a$, $a^1 = a$ etc.

Actually, we didn't talk much about a^1, because we generally just call it a, but, because it isn't as obvious as higher powers, the fact that $a^1 = a$ is often given as a sixth index law.

This definition of a^n can't be applied to zero, negative or fractional values of n, however: you can't multiply half an a together or -2 as together. We could define a^n where n is not a natural number in any way we like. But, of course, life is easier if we define it in such a way that the five index laws we met in A4-5 are still true when n is not a natural number.

If we define them as in Laws 6 to 10 below, then Laws 1 to 5 will still hold.

Law 6: $a^1 = a$

as explained above.

Law 7: $a^0 = 1$

We want to define a^0 such that the index laws still hold.

When we define it, the following must be true: $a^n \times a^0 = a^{n+0} = a^n$

So $a^n \times a^0$ must equal a^n

So a^0 must equal 1. In other words, anything to the power of 0 is 1.

Note: By this definition $0^0 = 1$. But 0 to the power of anything equals 0. To avoid this inconsistency, we leave 0^0 out of the definition and say $a^0 = 1$ if $a \neq 0$. 0^0 is undefined in the same way that $0 \div 0$ is undefined.

Practice

Q1 Evaluate without a calculator:

(a) 2^0	(b) 3^0	(c) p^0	(d) 1^0	(e) 72.9^0	(f) $(2x^2z)^0$
(g) 5^1	(h) 2.7^1	(i) s^1	(j) $(-1)^1$	(k) -1^1	(l) $(5x^2v)^1$

Law 8: $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$

By the index laws, $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$

So $a^{-n} \times a^n = 1$

So $a^{-n} = \frac{1}{a^n}$

Note that if a is a common fraction, then $\frac{1}{a}$ is that fraction turned upside down. For example, if $a = {}^3/_5$, then $\frac{1}{a} = {}^5/_3$. So $({}^3/_5)^{-4} = ({}^5/_3)^4$.

Note: By this definition $0^{-n} = \frac{1}{0^n}$. But 0 to the power of anything equals 0 and division by 0 is not defined. So Law 8 applies only if $a \neq 0$.

Practice

Q2 Evaluate without a calculator:

(a) 2^{-2}	(b) 10^{-3}	(c) 7^{-1}	(d) 1^{-8}
(e) 2^{-4}	(f) 0.5^{-2}	(g) 5^{-1}	(h) $(\frac{1}{2})^{-3}$
(i) 0.25^{-2}	(j) $({}^3/_2)^{-1}$	(k) $({}^4/_5)^{-2}$	(l) $({}^5/_2)^{-3}$

Q3 Write as fractions without negative indices:

(a) a^{-2}	(b) s^{-3}	(c) t^{-1}	(d) k^{-8}

(e) $2w^{-3}$ (f) $5v^{-1}$ (g) $(3p)^{-4}$ (h) $3(2ar)^{-5}$

(i) a^3x^{-3} (j) $2s^{-5}t^2$ (k) $2u^{-1}v^4w^{-3}$ (l) $\frac{1}{2}g^2h^{-5}p^{-3}$

Q4 Write these using negative indices rather than fractions:

(a) $\frac{1}{b^2}$ (b) $\frac{1}{e^3}$ (c) $\frac{1}{r^{10}}$ (d) $\frac{1}{a}$

(e) $\frac{2}{w^2}$ (f) $\frac{10}{h^5}$ (g) $\frac{1}{2b^2}$ (h) $\frac{3}{a^3}$

(i) $\frac{a^3}{c^2}$ (j) $\frac{4a^3}{tc^2}$ (k) $\frac{x^4}{4a^3r^2}$ (l) $4 \times \frac{3s^3}{6c^2}$

Q5 Simplify, giving the answers without negative indices:

(a) $a^6 \times a^{-2}$ (b) $c^{-4} \times c^3$ (c) $h^{-3} \times h^{-1}$ (d) $(r^{-3})^2$

(e) $t^{-3} \div t^5$ (f) $k^{-2} \div k^{-5} \times d^{-2}$ (g) $s^{-1} \times r^{-3} \times s^4$ (h) $(p^2 \div p^{-4})^{-3} \div p^3$

(i) $\frac{a^2v^{-3}x^{-4}}{4(v^3x^{-1})^{-2}} \div \frac{a^{-3}v^2x}{(2v^2x^{-1})^{-3}a}$

Law 9: $a^{1/n} = \sqrt[n]{a}$

$a^{1/2} \times a^{1/2} = a^{1/2+1/2} = a^1 = a$

$\therefore a^{1/2} \times a^{1/2} = a$

$\therefore a^{1/2} = \sqrt{a}$ . . . the square root of a, the number which, when squared, makes a.

In the same way,

$a^{1/3} \times a^{1/3} \times a^{1/3} = a^{1/3+1/3+1/3} = a^1 = a$

$\therefore a^{1/3} \times a^{1/3} \times a^{1/3} = a$

$\therefore a^{1/3} = \sqrt[3]{a}$ . . . the cube root of a, i.e. the number which, when cubed, makes a.

Likewise,

$a^{1/4} \times a^{1/4} \times a^{1/4} \times a^{1/4} = a^{1/4+1/4+1/4+1/4} = a^1 = a$

$\therefore a^{1/4} \times a^{1/4} \times a^{1/4} \times a^{1/4} = a$

$\therefore a^{1/4} = \sqrt[4]{a}$ . . . the fourth root of a, i.e. the number which, when raised to the power of 4, makes a.

In general,

$a^{1/n} = \sqrt[n]{a}$ . . . the n^{th} root of a, i.e. the number which, when raised to the power of n, makes a.

Law 10: $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

By the index laws, $a^{m/n} = a^{1/n \times m} = (a^{1/n})^m = (\sqrt[n]{a})^m$

In the same way, $a^{m/n} = a^{m \times 1/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$

So, for example $5^{3/4} = (\sqrt[4]{5})^3 = \sqrt[4]{5^3}$

Note that some roots of negative numbers aren't real numbers. Because of this, we restrict Laws 9 and 10 to positive values of a.

Note on irrational indices

We now have definitions for powers with any rational index. But we have not defined powers with irrational indices. It turns out that there is no real need to do that. So we don't.

$2^{\sqrt{2}}$ is undefined, as is 2^{π}, though, of course $2^{3.1415926}$ is defined. It is approximately 8.825.

Practice

Q6 Evaluate without a calculator:

(a) $4^{1/2}$ (b) $8^{1/3}$ (c) $16^{1/2}$ (d) $16^{0.5}$ (e) $32^{0.2}$

(f) $9^{-1/2}$ (g) $36^{-0.5}$ (h) $16^{-0.25}$ (i) $8^{2/3}$ (j) $9^{3/2}$

(k) $25^{1.5}$ (l) $32^{1.2}$ (m) $27^{-4/3}$ (n) $32^{-0.6}$ (o) $1024^{-0.3}$

(p) $4^{4.5}$ (q) $(-8)^{2/3}$ (r) $(-8)^{5/3}$ (s) $(-8)^{-2/3}$ (t) $9^{-1.5}$

Q7 Simplify:

(a) $u^{1/2} \times u^{5/2}$ (b) $u^{1/2} \times u^{1/3}$ (c) $h^{2/3} \div h^{-1/6}$

Q8 Write as single powers:

(a) $\sqrt{6}$ (b) $\sqrt[3]{10}$ (c) $\sqrt{5^3}$ (d) $\left(\sqrt{5}\right)^3$

(e) $2\sqrt{2}$ (f) $a\left(\sqrt[5]{a}\right)$ (g) $s \div \sqrt[3]{s}$ (h) $\sqrt{c} \div \sqrt[6]{c}$

Q9 Write in surd form (i.e. with roots):

(a) $5^{1/5}$ (b) $2^{-1/2}$ (c) $3^{0.25}$ (d) $7^{-0.2}$

(e) $\frac{1}{2^{-0.6}}$ (f) $w^{-2/3}$ (g) $x^{0.4}$ (h) $d^{-2.1}$

Summary of all 10 Index Laws

Below is a summary of the 10 index laws in a form which you can use to make sure you know them. Take a sheet of paper, cover one side of Laws 1 to 5 and write the expressions you have covered on the paper. Then do the same covering the other side. Then repeat for Laws 6 to 10.

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $a^m \div a^n = a^{m-n}$

Law 3: $(a^m)^n = a^{mn}$

Law 4: $(ab)^n = a^n b^n$

Law 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Law 6: $a^1 = a$

Law 7: $a^0 = 1$

Law 8: $a^{-n} = \frac{1}{a^n}$

Law 9: $a^{1/n} = \sqrt[n]{a}$

Law 10: $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Solve

Q51 If $x = y$, then $x^y = y^x$. Can you find any solutions to $x^y = y^x$ where $x \neq y$?

Q52 Sketch the graph of $y = x^x$ by hand. Then check by graphing it on a graphics calculator. Can you explain what you see on the calculator?

Revise

Revision Set 1

Q61 Evaluate without a calculator:

(a) 3^3 (b) 7^0 (c) 2^{-4} (d) $(\frac{1}{2})^{-3}$ (e) $27^{1/3}$

(f) $16^{-0.75}$ (g) $(-4)^2$ (h) $(-3)^3$ (i) 0.5^{-4} (j) $(-8)^{2/3}$

Q62 Write as single powers:

(a) $\sqrt[3]{10}$ (b) $\left(\sqrt{5}\right)^3$ (c) $3\sqrt{3}$ (d) $s \div \sqrt[3]{s}$

Q63 Write in surd form:

(a) $5^{1/5}$ (b) $2^{-\frac{1}{2}}$ (c) $\frac{1}{2^{-0.8}}$ (d) $d^{-2.1}$

Revision Set 2

Q71 Evaluate without a calculator:

(a) 2^3 (b) 4^0 (c) 10^{-4} (d) $(\frac{1}{2})^{-2}$ (e) $64^{1/3}$

(f) $16^{1.25}$ (g) $(-4)^2$ (h) -4^2 (i) 1^{-4} (j) $(-27)^{2/3}$

Q72 Write as single powers:

(a) $\sqrt[4]{12}$ (b) $\left(\sqrt{7}\right)^5$ (c) $5\sqrt[3]{5}$ (d) $c \div \sqrt[4]{c}$

Q73 Write in surd form:

(a) $2^{1/4}$ (b) $6^{-1/2}$ (c) $\dfrac{1}{2^{0.3}}$ (d) $x^{-1.5}$

Revision Set 3

Q81 Evaluate without a calculator:

(a) 3^4 (b) 1.4^0 (c) 3^{-3} (d) $\left(1/4\right)^{5/2}$ (e) $625^{0.75}$

(f) $32^{-0.6}$ (g) $(-6)^2$ (h) $(-4)^3$ (i) 0.2^{-2} (j) $(16)^{3/4}$

Q82 Write as single powers:

(a) $\sqrt{5}$ (b) $\left(\sqrt[4]{10}\right)^3$ (c) $5^3 \div \sqrt{5}$ (d) $s\sqrt[3]{s}$

Q83 Write in surd form:

(a) $2^{1/4}$ (b) $3^{-1/2}$ (c) $\dfrac{1}{9^{-0.6}}$ (d) $d^{0.75}$

Answers

Q1 (a) 1 (b) 1 (c) 1 (d) 1 (e) 1 (f) 1
(g) 5 (h) 2.7 (i) s (j) −1 (k) −1 (l) $5x^2v$

Q2 (a) ¼ (b) 0.001 (c) $1/7$ (d) 1 (e) $1/16$ (f) 4
(g) $1/5$ (h) 8 (i) 16 (j) $2/3$ (k) $25/16$ (l) $8/125$

Q3 (a) $\dfrac{1}{a^2}$ (b) $\dfrac{1}{s^3}$ (c) $\dfrac{1}{t}$ (d) $\dfrac{1}{k^8}$ (e) $\dfrac{2}{w^3}$ (f) $\dfrac{5}{v}$
(g) $\dfrac{1}{81p^4}$ (h) $\dfrac{3}{32a^5r^5}$ (i) $\dfrac{a^3}{x^3}$ (j) $\dfrac{2t^2}{s^5}$ (k) $\dfrac{2v^2}{uw^3}$ (l) $\dfrac{g^2}{2h^5p^3}$

Q4 (a) b^{-2} (b) e^{-3} (c) r^{-10} (d) a^{-1}
(e) $2w^{-2}$ (f) $10h^{-5}$ (g) $\frac{1}{2}b^{-2}$ (h) $3a^{-3}$
(i) a^3c^{-2} (j) $4a^3t^{-1}c^{-2}$ (k) $\frac{1}{4}x^4a^{-3}r^{-2}$ (l) $2s^3c^{-2}$

Q5 (a) a^4 (b) $\dfrac{1}{c}$ (c) $\dfrac{1}{h^4}$ (d) $\dfrac{1}{r^6}$ (e) $\dfrac{1}{t^8}$ (f) $\dfrac{k^3}{d^2}$
(g) $\dfrac{s^3}{r^3}$ (h) $\dfrac{1}{p^{21}}$ (i) $\dfrac{a^6}{32v^5x^4}$

Q6 (a) 2 (b) 2 (c) 4 (d) 4 (e) 2
(f) $1/3$ (g) $1/6$ (h) ½ (i) 4 (j) 27
(k) 125 (l) 64 (m) $1/81$ (n) $1/8$ (o) $1/8$
(p) 512 (q) 4 (r) −32 (s) ¼ (t) $1/27$

Q7 (a) u^3 (b) $u^{5/6}$ (c) $u^{5/6}$

Q8 (a) $6^{1/2}$ (b) $10^{1/3}$ (c) $5^{3/2}$ (d) $5^{3/2}$ (e) $2^{3/2}$
(f) $a^{6/5}$ (g) $s^{2/3}$ (h) $c^{5/6}$

Q9 (a) $\sqrt[5]{5}$ (b) $1/\sqrt{2}$ (c) $\sqrt[4]{3}$ (d) $1/\sqrt[5]{7}$ (e) $\sqrt[5]{2^3}$
(f) $1/\sqrt[3]{w^2}$ (g) $\sqrt[5]{x^2}$ (h) $1/\sqrt[10]{d^{21}}$

Q51 $x = 2$, $y = 4$ and vice versa. Are there others?

Q61	(a) 27	(b) 1	(c) $1/16$	(d) 8	(e) 3
	(f) $1/8$	(g) 16	(h) −27	(i) 16	(j) 4
Q62	(a) $10^{1/3}$	(b) $5^{3/2}$	(c) $3^{3/2}$	(d) $s^{2/3}$	
Q63	(a) $\sqrt[5]{5}$	(b) $1/\sqrt{2}$	(c) $\sqrt[5]{2^4}$	(d) $1/\sqrt[10]{d^{21}}$	
Q71	(a) 8	(b) 1	(c) 0.0001	(d) 4	(e) 4
	(f) 32	(g) 16	(h) −16	(i) 1	(j) 9
Q72	(a) $12^{1/4}$	(b) $7^{5/2}$	(c) $5^{4/3}$	(d) $c^{3/4}$	
Q73	(a) $\sqrt[4]{2}$	(b) $1/\sqrt{6}$	(c) $\frac{1}{\sqrt[10]{2^3}}$	(d) $1/\sqrt{x^3}$	
Q81	(a) 81	(b) 1	(c) $1/27$	(d) $1/32$	(e) 125
	(f) $1/8$	(g) 36	(h) −64	(i) 25	(j) 8
Q82	(a) $5^{1/2}$	(b) $10^{3/4}$	(c) $5^{5/2}$	(d) $s^{4/3}$	
Q83	(a) $\sqrt[4]{2}$	(b) $1/\sqrt{3}$	(c) $\sqrt[5]{9^3}$	(d) $\sqrt[4]{d^3}$	