M1 Maths

A5-12 Trigonometric Equations

- solving equations derived from trigonometric functions

Summary Learn Solve Revise Answers

Summary

Explicit trigonometric equations can be solved by substituting, using a calculator.

Implicit trig equations are solved, as with other equations types, by using inverse operations to undo the operations performed on the unknown, starting with the last one.

The trigonometric step can be undone using the inverse trig functions on a calculator. However, trig equations typically have more than one solution and the others need to be found using a unit circle.

The tan identity ($\tan\theta = \frac{\sin\theta}{\cos\theta}$) and the Pythagorean identity ($\sin^2\theta + \cos^2\theta = 1$) can be helpful in solving some trig equations.

Learn

As we know, a sinusoidal function is a function of the form $y = a \sin b\,(x + c) + d$ or $y = a \cos b\,(x + c) + d$. An example might be $y = 3 \sin 2\,(x - 5) + 8$.

Explicit Equations

If we substitute for the independent variable, x, we get an explicit equation. For instance, if $x = 30°$, we get $y = 3 \sin 2\,(30 - 10) + 8$. This can be evaluated using a calculator and comes to 9.93. That's not difficult. Just don't forget to make sure there's a bracket round the part that you want it to find the sine of, like this: $3 \sin (2 \times (30 - 10)) + 8$. Otherwise the calculator does sin 2, then multiplies by 20.

Practice

Q1 If $y = 2 \sin 10\,(x - 5) + 8$, find y when

(a) $x = 10$ (b) $x = 75$ (c) $x = 0$ (d) $x = -30$

Q2 If $y = 5 \sin 3(x + 15) - 4$, find y when

(a) $x = 20$ (b) $x = 250$ (c) $x = -10$ (d) $x = -300$

Q3 If $y = -\sin(x + 1) + 12$, find y when

(a) $x = 4$ (b) $x = 88$ (c) $x = 0$ (d) $x = -60$

Q4 The formula for tide height on a particular day at a particular place is $h = 1.6 \sin 30(t + 2.2) + 1.8$ where h is the height in metres and t is the time in hours since midnight. Find the tide height at :

(a) 4 p.m. (b) 11:24 a.m.

Q5 The height of a car on a Ferris wheel is given by $h = 8 \sin 3(t - 20) + 10$ where h is the height in metres and t is the time in seconds since starting. Find the height 3 min 18 s after starting.

Implicit Equations

If we substitute for the dependent variable, y, we get an implicit equation. For instance, if $y = 6.5$, we get

$$6.5 = 3 \sin(x - 10) + 8$$

Suppose we want to solve this getting the solutions between 0° and 360° inclusive ($0° \le x \le 360°$).

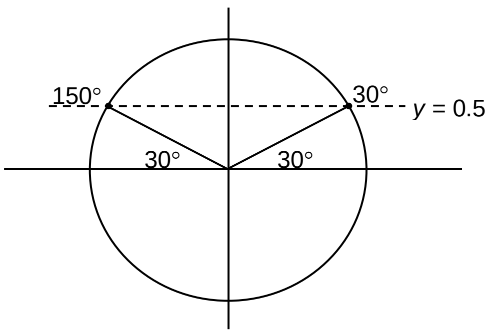
We do it the same way we solve any equation – by undoing the operations that were performed on x, starting with the last one.

The sequence of operations is: –10, sin, ×3, +8; to undo these we do: –8, ÷3, $\sin^{-1}$, +10.

$$6.5 = 3 \sin(x - 10) + 8$$

$-8 \rightarrow \quad -1.5 = 3 \sin(x - 10)$

$\div 3 \rightarrow \quad 0.5 = \sin(x - 10)$



$\sin^{-1} \rightarrow \quad 30° = x - 10 \quad$ or $\quad 150° = x - 10$

$+10 \rightarrow \quad x = 40° \quad$ or $\quad x = 160°$

The only difference from equations we've solved before is the $\sin^{-1}$ step with the two possible solutions. You learnt how to do this in Module M5-1 (Unit Circle and Trig Identities). Go back and refresh your memory if you need to.

Practice

Q6 Solve these equations, finding all the solutions for which $0 \le x \le 360°$.

(a) $6 = 4 \sin (x - 5) + 4$

(b) $\sin (x + 15) + 4.2 = 4.5$

(c) $10 \sin (x - 20) + 5 = 0$

(d) $3 \sin (x + 10) + 4 = 6$

(e) $19 - \sin x = 20$

(f) $27.2 = 4 \sin (x + 16) + 30.2$

(g) $\sin (x - 12) + 0.5 = 0$

(h) $2 \sin (x + 45) = -0.3$

Different Domains

Of course, there are a lot more solutions to the above equations outside the domain $0° \le x \le 360°$. Sometimes, we need solutions over a different domain like $0° \le x \le 720°$ or $-180° \le x \le 180°$.

When undoing the sin in the equation solving process, we have to make sure that we include all the angles within the required domain. If our required domain in the equation above had been $-360° \le x \le 720°$, then the $\sin^{-1}$ step would have given us:

$\sin^{-1}$ → $30° = x - 10$ or $150° = x - 10$

or $390° = x - 10$ or $510° = x - 10$

or $-330° = x - 10$ or $-210° = x - 10$

[Note that we can get all those outside of $0° \le x \le 360°$ by adding or subtracting multiples of 360° to those inside the domain.]

Then the last step of adding 10 would give us

+10 → $x = 40°$ or $x = 160°$ or $x = 400°$ or $x = 520°$ or $x = -320°$ or $x = -200°$

Practice

Q7 Solve the following equations over the domain $0° \le x \le 720°$.

(a) $9 = 2 \sin (x - 5) + 8$

(b) $4 \sin (x - 5) + 4 = 3$

Q8 Solve the following equations over the domain $-360° \le x \le 360°$.

(a) $10 \sin (x + 10) - 3 = 4$

(b) $5 \sin (x + 10) + 3 = 2$

Q9 Solve the following equations over the domain $-180° \le x \le 180°$.

(a) $20 \sin (x + 12) = 16$

(b) $5 = 4 \sin (x + 9) + 6$

(c) $\sin (x - 2) - 0.5 = 0$

(d) $4 \sin (x + 40) + 5 = 1$

Solutions Moving In and Out of the Domain

We also need to be careful in the last step of the equation solving (adding 10 in the example above) that none of our solutions move outside the required domain. For instance, if the last step had been subtracting 40 instead of adding 10, then the –320° solution would have been –370° instead and so shouldn't be included in our answer. However, if a solution moves out of the domain, then another one generally moves in at the other end. The width of our domain is 3 lots of 360°. If we add 3 lots of 360° to –370°, we get 710°, which is inside the required domain, and so should be included.

Our final solution would then be

$x = -10°$ or $x = 110°$ or $x = 350°$ or $x = 470°$ or $x = 710°$ or $x = -250°$

Practice

Q10 Solve these equations, finding all the solutions for which $0 \le x \le 360°$.

(a) $7 = 2 \sin (x + 60) + 6$

(b) $2 \sin (x - 45) + 4 = 3$

(c) $10 \sin (x + 30) - 1 = 4$

(d) $3 \sin (x + 420) + 2 = 1$

(e) $4 \sin (x + 30) = 16$

(f) $3 = 4 \sin (x - 55) + 5$

(g) $\sin (x - 10) + 0.1 = 0$

(h) $4 \sin (x - 1000) + 5 = 1$

If the parameter b is greater than 1

You may have noticed that, so far, you haven't been asked to solve any problems like $10 \sin 2 (x + 10) - 3 = 4$ where the parameter b is not equal to 1. In this case it is 2.

Using what we know so far, we would probably solve this equation for $0° \le x \le 360°$ like this:

$10 \sin 2 (x + 30) - 3 = 4$

$+3 \rightarrow$ $10 \sin 2 (x + 30) = 7$

$\div 10 \rightarrow$ $\sin 2 (x + 30) = 0.7$

$\sin^{-1} \rightarrow$ $2 (x + 30) = 44°$ or $2 (x + 30) = 136°$

$\div 2 \rightarrow$ $x + 30 = 44°$ or $x + 30 = 136°$

$-30 \rightarrow$ $x = 14°$ or $x = 106°$

But there are yet more solutions between 0 and 360°.

Consider the ÷2 step. We need all the solutions that lie between 0° and 360° after we divide by 2. This means that we have to include all the solutions that lie between 0° and 720° before we divide by 2.

In other words, when we do the $\sin^{-1}$ step, we have to find all the solutions that lie between 0 and 720°. The two we got at this step were 44° and 136°. These are both on the first time around the circle. There will be solutions on the second time around the circle too. They are 360° more than those on the first time round, i.e. 404° and 496°.

Once we do the next step of dividing by 2, these two extra solutions will become 202° and 248°, both of which are within the domain we are interested in.

So, if the parameter b in the equation, is 2, then, at the $\sin^{-1}$ step, we have to find solutions on the first 2 times around the circle.

Similarly, if $b = 3$, then at the $\sin^{-1}$ step, we have to find solutions on the first 3 times around the circle. The solutions on the third time around will be 720° more than those on the first time around.

And so on if $b > 3$. If b is not a whole number, round it up to be on the safe side.

The complete solution to $10 \sin 2(x + 30) - 3 = 4$ for $0° \le x \le 360°$ will be as follows:

$$10 \sin 2(x + 30) - 3 = 4$$

$+3 \rightarrow$ $\quad 10 \sin 2(x + 30) = 7$

$\div 10 \rightarrow$ $\quad \sin 2(x + 30) = 0.7$

$\sin^{-1} \rightarrow$ $\quad 2(x + 30) = 44°$ or $2(x + 30) = 136°$ or $x + 30 = 202°$ or $x + 30 = 248°$

$-30 \rightarrow$ $\quad x = 14°$ or $x = 106°$ or $x = 172°$ or $x = 218°$

Practice

Q11 Solve these equations, finding all the solutions for which $0° \le x \le 360°$.

(a) $6 = 4 \sin 2(x - 5) + 4$

(b) $\sin 3(x + 15) + 4.2 = 4.5$

(c) $10 \sin 2(x - 30) + 5 = 0$

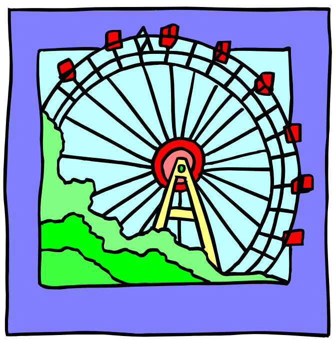
(d) $3 \sin (x + 10) + 4 = 6$

(e) $19 - 2 \sin 5x = 20$

(f) $27.2 = 4 \sin 3(x + 16) + 30.2$

(g) $\sin 1.5(x - 12) - 0.5 = 0$

(h) $2 \sin 2.5(x + 45) = \sqrt{3}$



Equations with Cosines

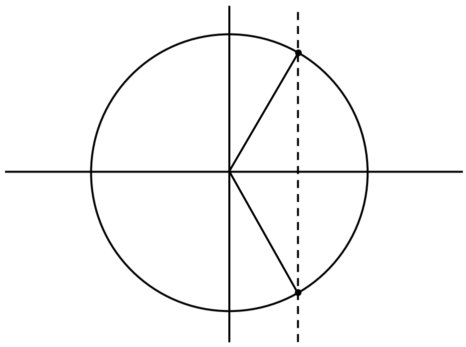
All the above equations have involved sines. Equations with cosines work just the same way. The only difference is the circle diagram.

Here is an example.

$10 \cos 2\,(x - 40) + 5 = 10$

$-5 \rightarrow \quad 10 \cos 2\,(x - 40) = 5$

$\div 10 \rightarrow \quad \cos 2\,(x - 40) = 0.5$



$\cos^{-1} \rightarrow \quad 2\,(x - 40) = 60°$ or $2\,(x - 40) = 300°$ or $2\,(x - 40) = 420°$ or $2\,(x - 40) = 660°$

$\div 2 \rightarrow \quad x - 40 = 30°$ or $x - 40 = 150°$ or $x - 40 = 210°$ or $x - 40 = 330°$

$+40 \rightarrow \quad x = 70°$ or $x = 190°$ or $x = 250°$ or $x = 370°$

The last solution has to be changed to $x = 10°$ to bring it inside the required range. So the final solution is :

$x = 10°$ or $x = 70°$ or $x = 190°$ or $x = 250°$

Practice

Q12 Solve these equations, finding all the solutions for which $0 \le x \le 360°$.

(a) $3 = 4 \cos (x - 5) + 2$

(b) $\cos 2\,(x + 10) + 4 = 4.5$

(c) $10 \cos 2\,(x - 40) - 5\sqrt{2} = 0$

(d) $6 \cos (x + 10) + 4 = 6$

(e) $12 + 4 \cos 3\,x = 10$

(f) $27.2 = 4 \cos (x + 20) + 27.9$

(g) $\cos 1.5\,(x - 2) - 0.5 = 0$

(h) $4 \cos 0.8\,(x + 60) = -2$

Equations derived from sinusoidal functions are often called trigonometric equations. Here are some real-life problems which require the solution of trigonometric equations.

Practice

Q13 The height of a car on a Ferris wheel is given by $h = 8 \sin 2(t - 20) + 10$ where h is the height in metres and t is the time in seconds since starting. Find the times in the first 360 s when the height is 14 m.

Q14 The height of a car on a Ferris wheel is given by $h = 12 \sin 3(t + 30) + 12$ where h is the height in metres and t is the time in seconds since starting. Find the times in the first revolution when the height is 13 m.

Q15 The formula for the height of the tide one day is $h = 1.2 \sin 28(t - 3.2) + 1.6$ where h is the height in metres and t is the time in hours since midnight. At what times during that day is the tide height 0.5 m?

Q16 A Ferris wheel has radius 8 m and its axel is 9 m above the ground. If it rotates once every 180 seconds, what fraction of the time does a point on the circumference of the wheel spend more than 12 m above the ground? What if it rotates once every 90 seconds?

Q17 The voltage of an AC power source is given by the formula $V = 315 \cos 18000t$, where V is the voltage in volts and t is the time in seconds.

(a) what is the peak voltage?

(b) what is the frequency?

(c) how long in each cycle does the voltage exceed +250V?

Q18 The height of the tide on Tuesday is given by the function $h = 1.5 \sin 30(t + 2) + 1.8$ where h is the height in metres and t is the time in hours since midnight.

(a) Find the tide height at 5 a.m.

(b) Find the times of the high and low tides on Tuesday.

(c) Find the times during Tuesday when the tide height is 0.7 m.

(d) A causeway is traversable only when the tide height is below 1 m. For how long on Tuesday will the causeway be traversable?

(e) For what percentage of the time on Tuesday will the tide be within 0.2 m of the high tide level?

Q19 The temperature follows a sinusoidal pattern over time ranging from 10° to 24° each day with the highest temperature occurring at 3 pm.

(a) Write a formula for the temperature, T at any number of hours, t, since midnight.

(b) What is the temperature at 11 a.m.?

(c) For how long each night is the temperature below 12°?

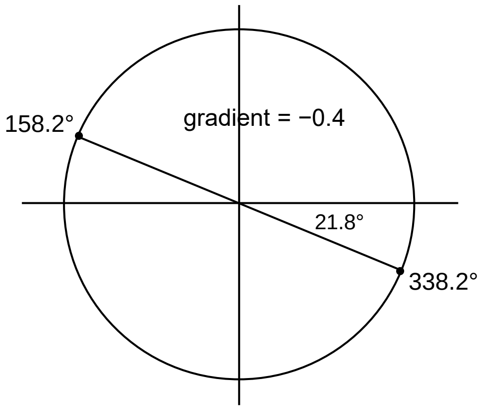
Q20 The water depth at a dock varies according to the formula $w = 1.1 \cos 30(t - 2) + 2.4$, where w is the water depth in metres and t is the number of hours since midnight. A fishing boat needs to dock and unload, but can only be in the dock when the water depth is more than 3 m. If it arrives at 6 a.m., when will it first be able to dock and how long can it stay?

Equations with Tans

Equations with tangents occur less frequently than those with sines and cosines. But they are solved pretty much the same way (except of course that we use the gradient of OP on the unit circle rather than the y- or x-coordinate of P – you should remember this from Module M5-1.)

For example, suppose we have $5 \tan (2x - 15) + 12 = 10$ for $0° \le x \le 360°$.
This is solved like this:

$$5 \tan (2x - 15) = -2$$
$$\tan (2x - 15) = -0.4$$
$$2x - 15 = \tan^{-1} -0.4$$
$$= -22°$$



So $2x - 15 = 158$ or $2x - 15 = 338$

$2x = 173$ or $2x = 353$

$x = 87°$ or $x = 177°$

Practice

Q21 Solve these equations, finding all the solutions for which $0 \le x \le 360°$.

(a) $\tan x = 2.1$

(b) $3 \tan x = -0.88$

(c) $4 \tan 2 (x - 30) + 5 = 13$

(d) $5 \tan (x + 10) - 1 = -6$

Quadratic Trig Equations

Consider the equation $\sin^2\theta - 3\sin\theta + 2 = 0$.

[Remember that $\sin^2\theta$ is a shorthand way of writing $(\sin\theta)^2$.]

This is a quadratic with $\sin\theta$ as the unknown. It can be solved by factorising:

$$(\sin\theta - 1)(\sin\theta - 2) = 0$$

Then $\sin\theta = 1$ or $\sin\theta = 2$

Using the unit circle (or just picturing it), $\sin\theta = 1$ gives the solution $\theta = 90°$

$\sin\theta = 2$ has no solutions.

So the only solution is $\theta = 90°$

Practice

Q22 Solve these equations, finding all the solutions for which $0 \le x \le 360°$.

(a) $\sin^2\theta + \sin\theta - 2 = 0$

(b) $2\cos^2\theta - \cos\theta - 1 = 0$

Note there will be 4 solutions to (b)

(c) $\tan^2\theta - 2\tan\theta - 3 = 0$

(d) $3\sin^2\theta = 2 - 5\sin\theta$

Equations using the tan and Pythagorean Identities

Tan identity

Consider the equation $\sin\theta = 2\cos\theta$.

This can be solved using the $\tan\theta$ identity.

Divide both sides by $\cos\theta$. This gives $\dfrac{\sin\theta}{\cos\theta} = 2$

Which, using the tan identity, is $\tan\theta = 2$

This can be solved as before.

Pythagorean Identity

Consider $2\cos^2\theta = 7\sin\theta + 5$.

This looks a bit like a quadratic, but contains sin and cos, therefore two different variables.

But we can get around this by using the Pythagorean identity, $\cos^2\theta = 1 - \sin^2\theta$.

Subbing for $\cos^2\theta$, we get $2(1 - \sin^2\theta) = 7\sin\theta + 5$

Then $2 - 2\sin^2\theta = 7\sin\theta + 5$

Then $0 = 2\sin^2\theta + 7\sin\theta + 3$

This is then solvable as a quadratic as above.

Practice

Q23 Solve these equations, finding all the solutions for which $0 \le \theta \le 360°$.

(a) $\cos\theta = 3\sin\theta$ (b) $\cos\theta + \sin\theta = 0$

(c) $\sin^2\theta - \cos\theta + 1 = 0$ (d) $2\sin^2\theta + 3\cos\theta + 3 = 0$

(e) $2\cos^2\theta = 7\sin\theta + 5$ (f) $6\cos^2\theta + \sin\theta = 7$

Solve

Q51 Solve $2\sin^4 x = ½$, finding all the solutions for which $-360° \le \theta \le 720°$.

Q52 Using n to stand for any integer, find expressions for all the solutions to:

(a) $\sin x = 0.5$ (b) $3\tan^2 2x = 1$

Q53 Each day, John's mood oscillates according to the formula $j = 5\sin 30(t + 3) + 5$, where j is his mood index and t is the time in hours since midnight. His wife Betty's mood index is given by $b = 4\cos 15(t - 4) + 5$. When both have a mood index below 2, they fight. How many hours and minutes a day do they fight?

Revise

Revision Set 1

Q61 If $y = -5\sin(2x + 20) + 12$, find y when $x = 4$

Q62 The formula for tide height on a Tuesday at Squid Beach is $h = 2.4\sin 30(t + 4.5) + 2.8$ where h is the height in metres and t is the time in hours since midnight. Find the tide height at 5 p.m.

Q63 Solve the following, finding all the solutions for which $0 \le \theta \le 360°$.

(a) $8\cos(x + 20) + 11 = 14$ (b) $4\sin 3(x - 10) - 5 = -2$

(c) $\tan 2(x + 45) + 1 = 2.6$ (d) $\sin^2\theta - 6 = 5\sin\theta$

(e) $2\cos^2\theta = 7\sin\theta + 5$ (f) $\cos\theta + 2\sin\theta = 0$

Q64 The water depth at a sand bar varies according to the formula $w = 1.5\cos 30(t + 4) + 2.2$, where t is the number of hours midnight.
A fishing boat needs to cross the sand bar, but can only do so when the water

depth is more than 3 m. What is the earliest in the morning that it can cross and when will it last be able to cross back if it needs to be back before midday?

Answers

Q1 (a) 9.53 (b) 7.32 (c) 6.47 (d) 8.35
Q2 (a) 0.83 (b) 0.83 (c) −2.71 (d) −7.53
Q3 (a) 11.91 (b) 11.00 (c) 11.98 (d) 12.86
Q4 (a) 1.63 m (b) 2.99 m
Q5 10.84 m
Q6 (a) 35° or 155° (b) 2° or 148° (c) 230° or 350° (d) 32° or 128°
(e) 270° (f) 33° or 115° (g) 222° or 342° (h) 144° or 306°
Q7 (a) 35°, 155°, 395°, 515° (b) 21°, 36°
Q8 (a) 34°, 126°, –326°, –234° (b) –12°, –168°, 192°, 348°
Q9 (a) 31°, 115° (b) –23°, –175°
(c) 32°, 152° (d) –130°
Q10 (a) 90°, 330° (b) 255°, 15°
(c) 0°, 120°, 360° (d) 139°, 281°
(e) 136°, 344° (f) 265°, 25°
(g) 196°, 4° (h) 190°
Q11 (a) 20°, 80°, 200°, 260° (b) 36°, 111°, 156°, 231°, 276°, 351°
(c) 15°, 135°, 195°, 315° (d) 31.8°, 137.2°
(e) 42°, 66°, 114°, 138°, 186°, 210°, 258°, 302°, 330°, 354°
(f) 60°, 98°, 180°, 208°, 300°, 328°
(g) 32°, 112°, 272°, 352° (h) 9°, 117°, 193°, 261°, 297°, 333°
Q12 (a) 73°, 297° (b) 20°, 140°, 200°, 320°
(c) 17.5°, 62.5°, 197.5°, 242.5° (d) 60.5°, 279.5°
(e) 40°, 80°, 160°, 200°, 280°, 320° (f) 80°, 240°
(g) 42°, 202°, 282° (h) 90°, 240°
Q13 35 s, 95 s, 205 s, 275 s
Q14 58.4 s, 118.4 s
Q15 5:34 a.m., 7:15 a.m., 6:26 p.m., 8:07 p.m.
Q16 0.38, same
Q17 (a) 315 V (b) 50 Hz (c) 0.0039 s
Q18 (a) 1.05 m (b) High: 1 a.m., 1 p.m.; Low: 7 a.m., 7 p.m.
(c) 5:34 a.m., 8:26 a.m., 5:34 p.m., 8:26 p.m. (d) 7 h 42 min (e) 16.6%
Q19 (a) $T = 7\cos 15(t + 9) + 17$ (b) 20.5° (c) 5 h 55 min
Q20 12:06 p.m., 3 h 48 min
Q21 (a) 64.5°, 244.5° (b) 163.7°, 343.7° (c) 61.7°, 151.7°, 241.7°, 331.7° (d) 125°, 305°
Q22 (a) 90° (b) 0°, 120°, 240°, 360° (c) 71.6°, 135°, 251.6°, 315° (d) 19.5°, 160.5°
Q23 (a) 18.4°, 198.4° (b) 135°, 315° (c) 0°, 360°
(d) 120°, 240° (e) 240°, 330° (f) 30°, 150°, 199.5°, 340.5°

Q51 $x = -315°, -225°, 45°, 135°, 405°, 495°$
Q52 (a) One possible answer: $x = 45° + 360n°$ or $x = 135° + 360n°$
(b) One possible answer: $x = 180n° \pm 30°$
Q53 2 h 32 min

Q61 9.65
Q62 0.48 m
Q63 (a) 68°, 292° (b) 6°, 33°, 126°, 154°, 246°, 274° (c) 74°, 164°, 254°, 344°

(d) 270° (e) 225°, 315° (f) 153°, 333°

Q64 6:04 a.m., 9:56 a.m.