M1 Maths

A5-11 Trigonometric Functions

- general form and graph shape of sinusoidal functions
- applications of sinusoidal functions
- drawing graphs of sinusoidal functions from the formulas and vice versa
- non-sinusoidal trigonometric functions

Summary Learn Solve Revise Answers

Summary

Sinusoidal functions have the general form $y = a \sin b(x + c) + d$ or $y = a \cos b(x + c) + d$ with four parameters, a, b, c and d.

The graph is a wave with the following characteristics: amplitude a, period $^{360}/_{b}$, phase shift $-c$ and mean position d.

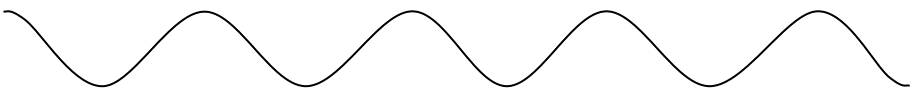
To find the formula for a given graph, determine the characteristics, calculate the parameters and write the formula. To sketch a graph from the formula, read off the parameters, calculate the characteristics, then draw the graph. A procedure for doing this easily is detailed in this module.

Other trigonometric functions have periodic (repeating) graphs, but of a different shape from sinusoidal graphs.

Learn

Wave Functions

There are many relations whose graphs are waves like this:



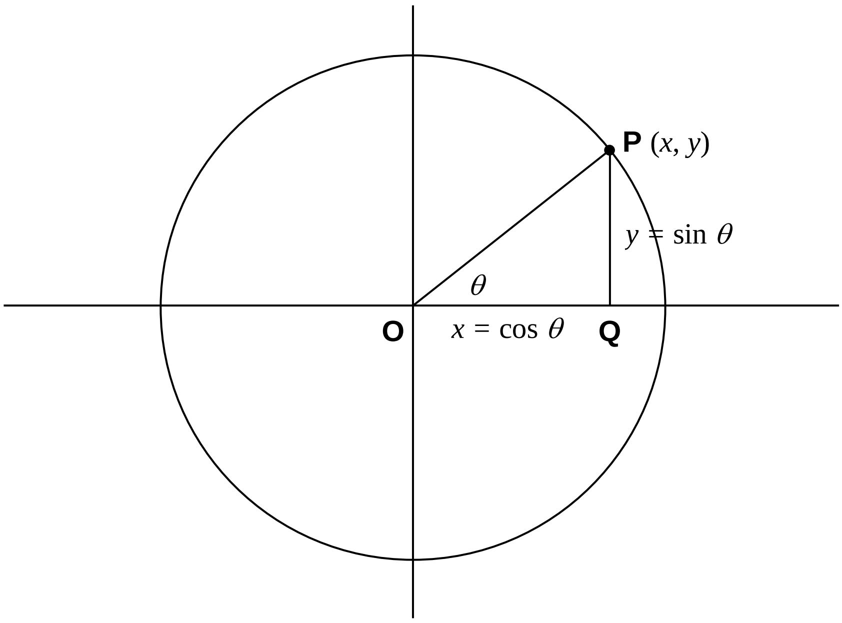
Some examples are:

- the relation between water height and time as waves pass a given point
- the relation between water height and position in a photograph of waves on water
- air pressure vs time as sound waves pass
- the height of a weight bouncing on the end of a spring vs time

- tide height vs time
- AC voltage vs time
- how far the sun is north of the equator through the year
- mean temperature vs time through the year at a given location
- the height of a Ferris wheel car vs time

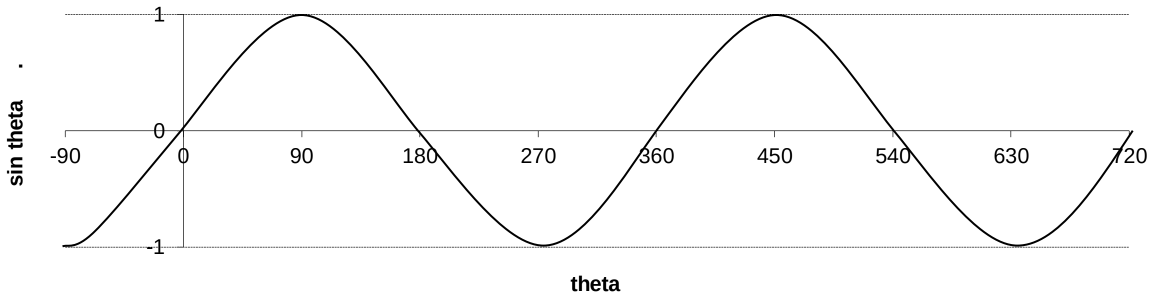
Relations which graph as waves can be modelled with the function $y = \sin\theta$.

You have seen that $\sin\theta$ can be defined in terms of the unit circle like this:



When $\theta = 0$, $\sin\theta = 0$; as θ increases to 90°, $\sin\theta$ increases to 1; as θ increases to 180°, $\sin\theta$ decreases back to 0, as θ approaches 270°, $\sin\theta$ approaches –1; and so on.

If we draw a graph of $\sin\theta$ vs θ, the graph would look like this.

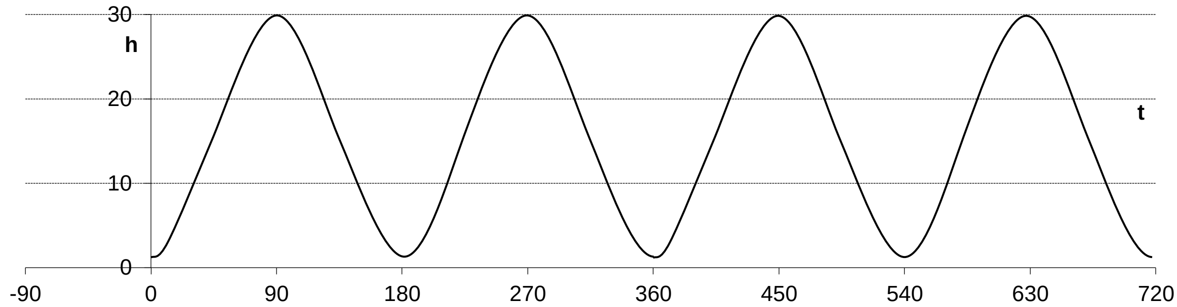


This graph would continue as a never-ending series of waves to the left and the right.

Variations

Suppose we wanted to model the relation between the height of the Ferris wheel car and time. It would be a wave quite like the one above. The function $h = \sin t$ (the graph above with h (height in metres) and t (time in seconds) as the variables instead of y and θ) would do the job – as long as the top of the ferris wheel was only 1 m above the ground, the bottom was 1 m under the ground and it took 360 seconds to go around!

If the car had a maximum height of 30 m, a minimum height of 2m and went round once every 180 s and started at the lowest point at $t = 0$, then we would need a different function – the one below.



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To get this graph we need to change four things: the mean position, the amplitude, the period, and the phase shift. These four things are called the characteristics of the graph and by setting them correctly we can make a wave that will fit any of the situations listed at the start of this module.

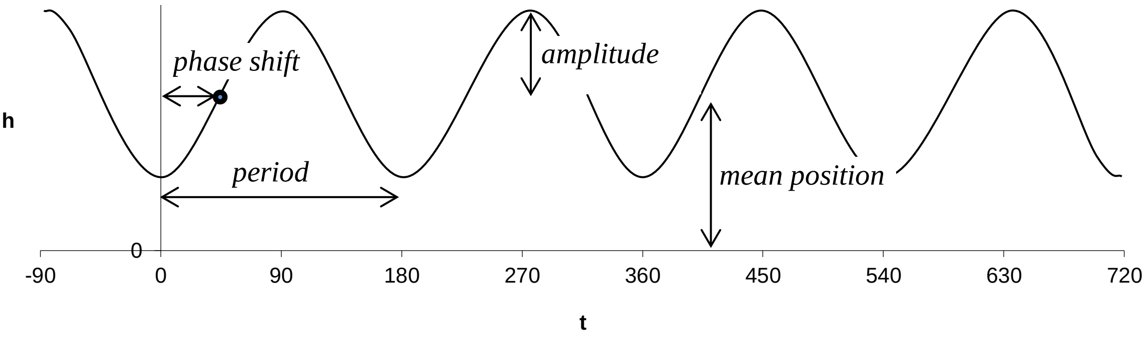
Mean position (sometimes called vertical shift) – this is the mean of the highest and lowest values the function takes. In the Ferris wheel example, it is $(2 + 30) \div 2$ which is 16. It is the mean height of the Ferris wheel car.

Amplitude – this is how far the value of the function gets above or below the mean position. In other words it is the maximum value minus the mean position. In the Ferris wheel example, it is 30 – 16 which is 14. It is also half the difference between the highest and lowest values. This is $(30 - 2) \div 2$ which is 14.

Period – this is the number of units along the x-axis before the graph repeats. In the graph of $y = \sin x$, the period is 360. In the Ferris wheel example it is 180.

[We sometimes specify the frequency instead of the period. The frequency is the reciprocal of the period, i.e. $1 \div$ period. It is the number of cycles per second rather than the number of seconds per cycle. For the Ferris wheel, the frequency is $^1/_{180}$.]

Phase shift – this is how many units the graph has been moved to the right relative to the graph of $y = \sin x$. To find the phase shift, find a point on the graph where it is at the mean position and rising (the mar point – mar stands for mean and rising). It is marked with a dot on the graph below. The x-coordinate (in this case t-coordinate) of the mar point is the phase shift. In the Ferris wheel example, this is +45. So the phase shift is 45.



Finding the formula for a graph

How do we find the formula for this function? Wave functions with various mean positions, amplitudes, periods and phase shifts are called sinusoidal functions. ‘Sinusoidal’ is just the adjective from ‘sine’. Sinusoidal functions have the general form

$$y = a \sin b(x + c) + d$$

They contain the sine function and four parameters, a, b, c and d.

Once we know the characteristics of the wave, the values of the parameters are easy to work out.

a is equal to the amplitude

b is equal to 360 divided by the period

c is the negative of the phase shift

d is equal to the mean position.

In the Ferris wheel example, $a = 14$, $b = 360 \div 180 = 2$, $c = -45$ and $d = 16$ and the formula is

$$h = 14 \sin 2(t - 45) + 16$$

So, if we have a sinusoidal graph and need to find the formula, we read off the characteristics, calculate the parameters using the relationships in red above,then substitute them into the general form to get the formula.

graph → characteristics → parameters → formula

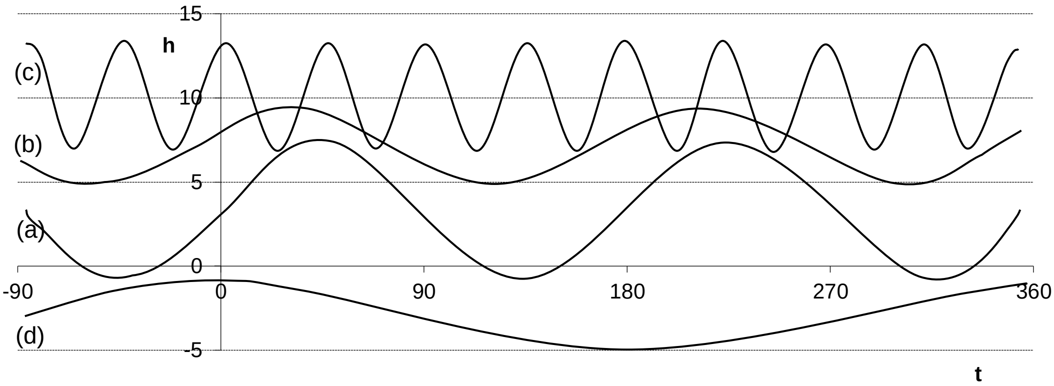
Practice

Q1 For each of the functions below,

(i) give the characteristics: mean position, amplitude, period and phase shift,

(ii) then find the values of the parameters a, b, c and d,

(iii) then find the formula.

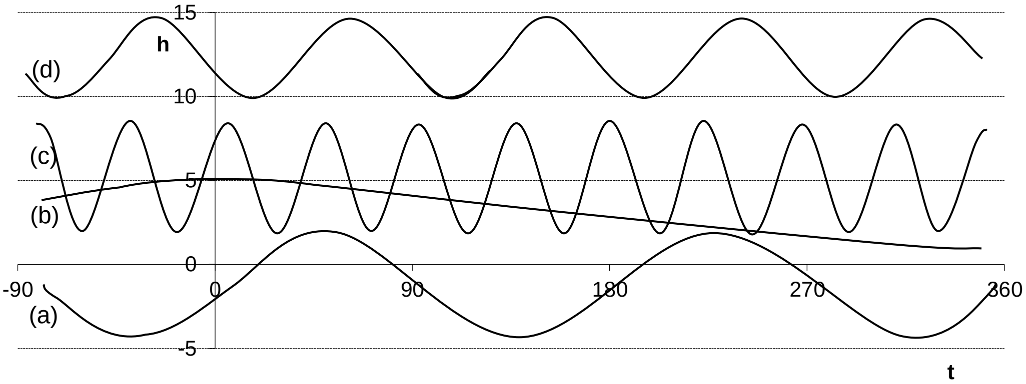


Q2 For each of functions below,

(i) give the characteristics: mean position, amplitude, period and phase shift,

(ii) then find the values of the parameters a, b, c and d,

(iii) then find the formula.



Drawing the graph given the formula

To draw the graph for a given formula, we use the reverse process: we read the parameters from the formula, calculate the characteristics, then use them to sketch the graph.

formula → parameters → characteristics → graph

To go in reverse, we need

the amplitude = a

the period = 360 ÷ b

the phase shift = $-c$

the mean position = d.

These should be fairly obvious from the forward versions

Tips for sketching a good graph

Suppose you are graphing $h = 14 \sin 2(t - 45) + 16$ (the Ferris wheel function from before).

First you work out the characteristics:

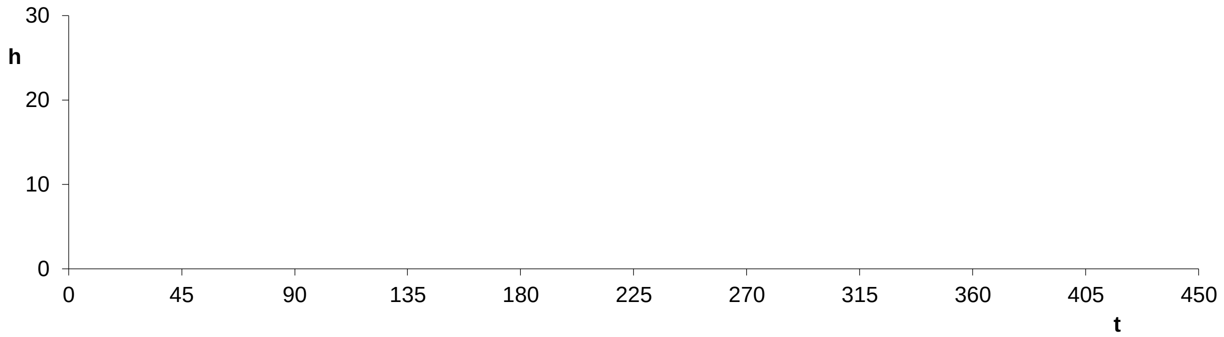
mean position = 16

amplitude = 14

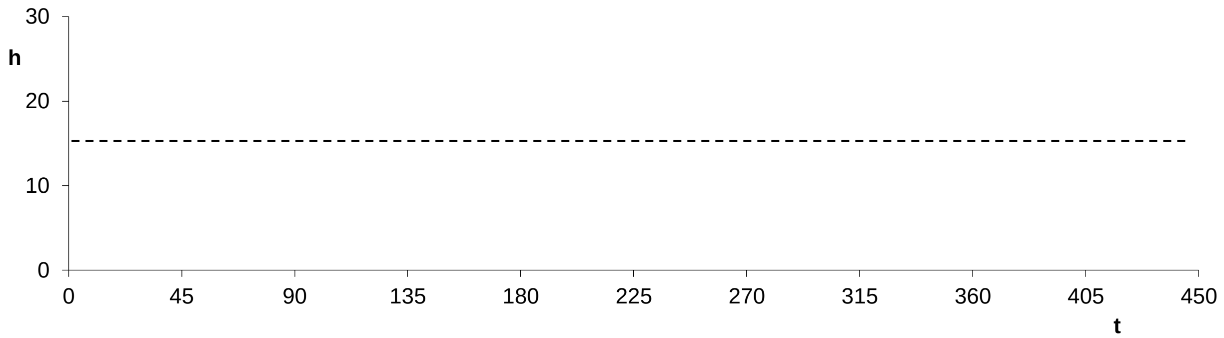
period = 360 ÷2 = 180

phase shift = 45

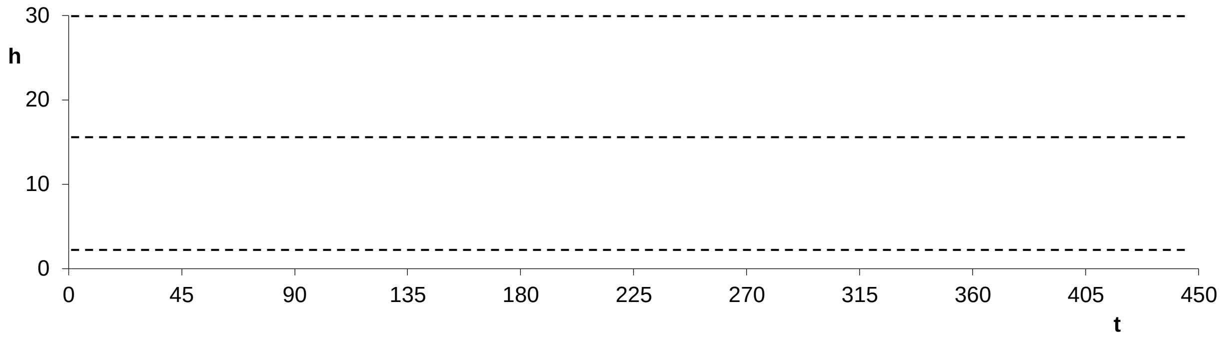
Then draw axes with suitable ranges. Generally try to include a bit over two complete cycles (periods), so in this case we will make the horizontal axis go from 0 to 450.



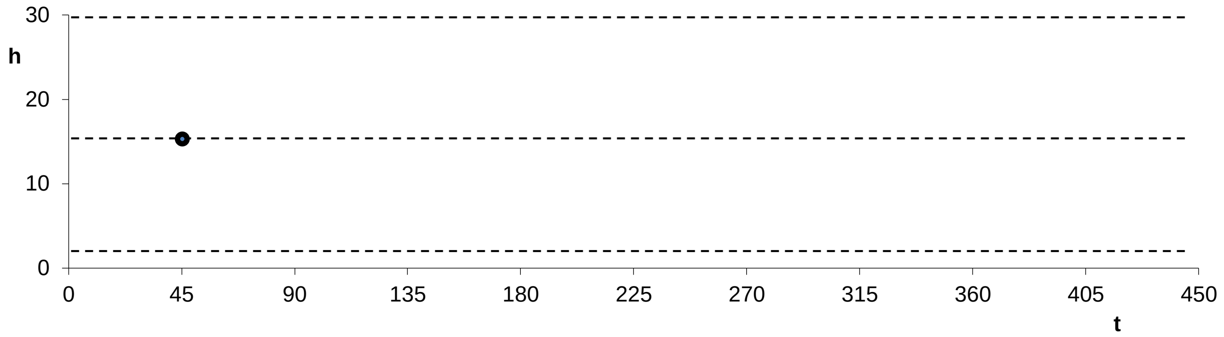
Once you have the axes drawn, draw a horizontal dotted or pencil line at the mean position.



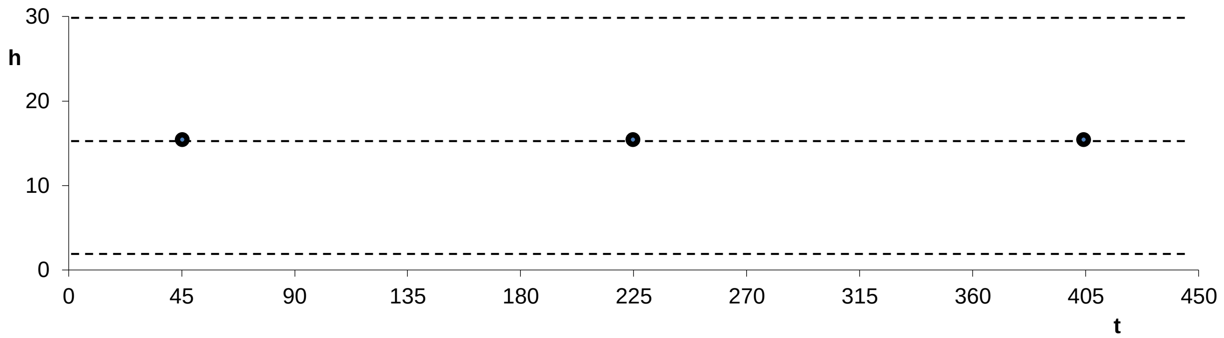
Then draw dotted or pencil lines a units above and below the mean position. These will be the highest and lowest levels of the graph.



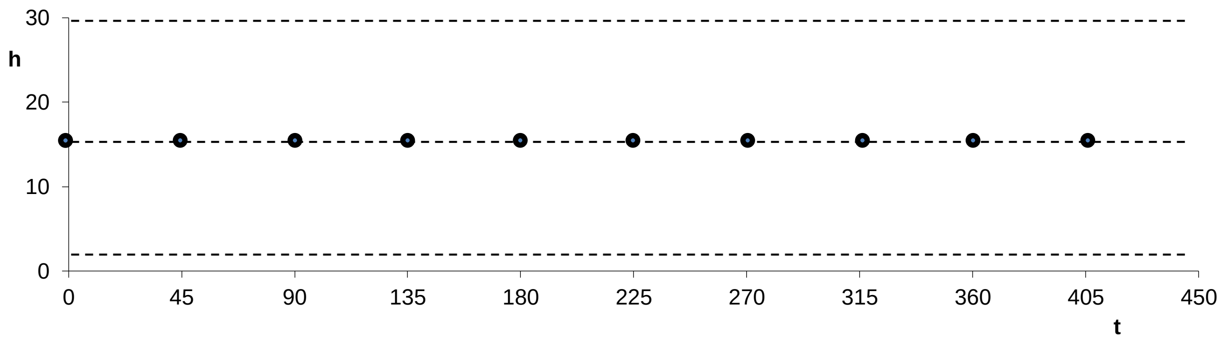
Then mark the phase shift by placing the mar point on the mean position line at an x-value equal to the phase shift.



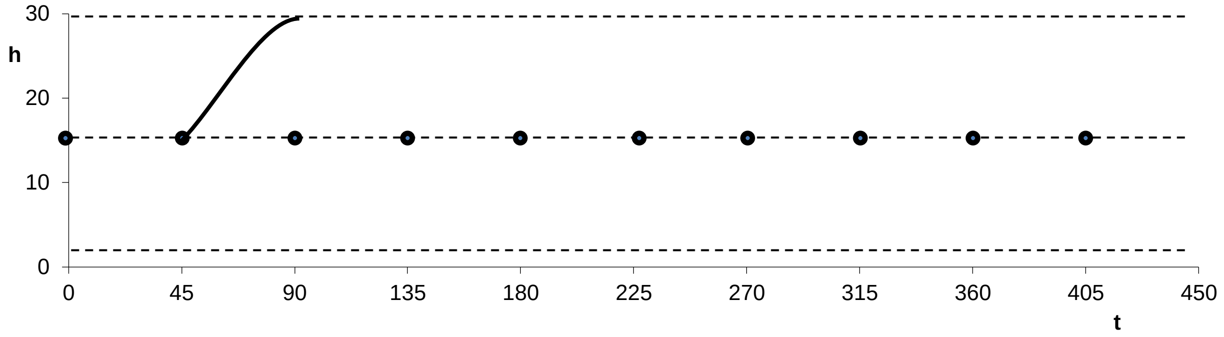
Then mark one period further along the mean position line, then another period further along still until you reach the end of the graph.



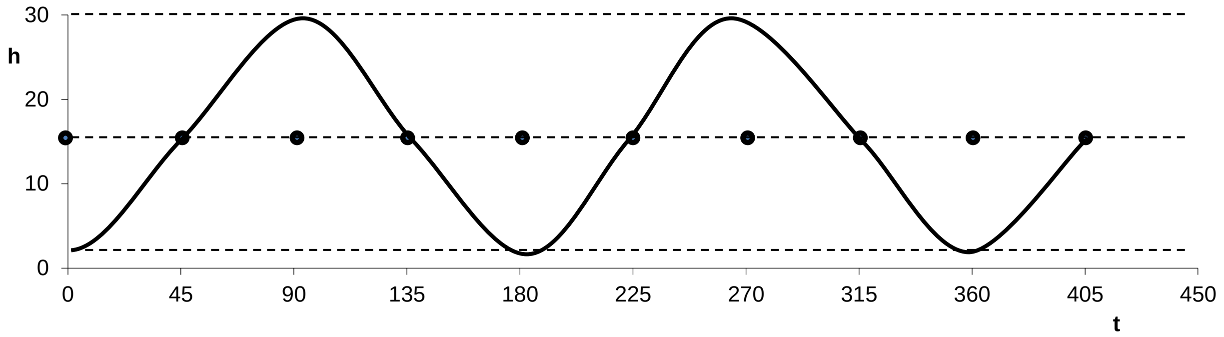
Then divide the periods into quarters by putting 3 more marks evenly spaced along the mean position line between each of the existing marks.



Then, starting at the phase shift mark, draw a quarter cycle upwards from the mean position line to the top line.



Then draw the next quarter cycle and so on.



Practice

Q3 Without your calculator, sketch graphs of the following sinusoidal functions. Then use your calculator to check your sketch.

(a) $y = \sin x$ (b) $y = 2 \sin (x - 90) + 4$

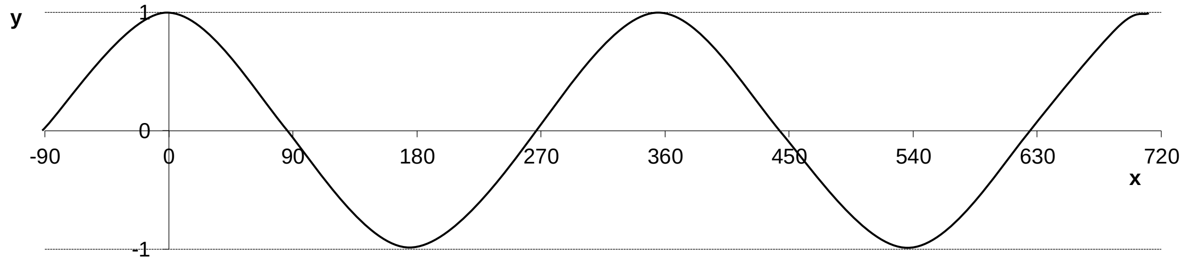
(c) $y = 5 \sin 2(x - 45) + 5$ (d) $y = 0.5 \sin 4(x + 10) - 2$

(e) $y = 3 \sin 4(x + 5) + 17$ (f) $y = 5 \sin \frac{1}{2}(x - 180)$

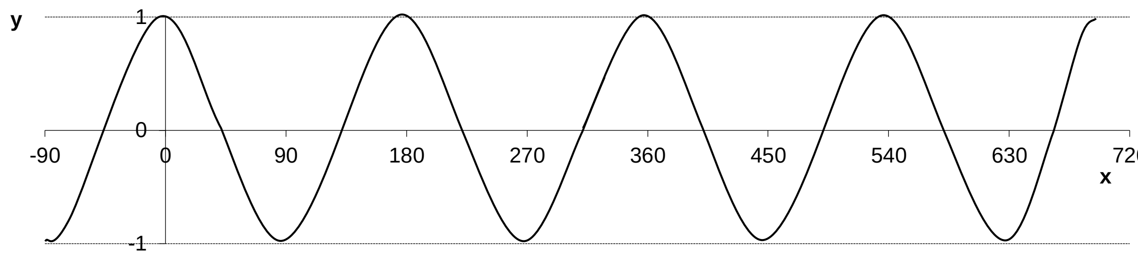
(g) $y = \sin 5(x - 30) - 2$ (h) $y = 2 \sin 2x + 3$

Using cosines and negatives for sinusoidal functions

The graph of $y = \cos x$ looks like this.



The only difference between this and the sine function is the phase shift. This can also be written as $y = \sin(x + 90)$. In fact any sine function can be written as a cosine function and any cosine function can be written as a sine function. The only thing that will change is the phase shift and thus the value of c.

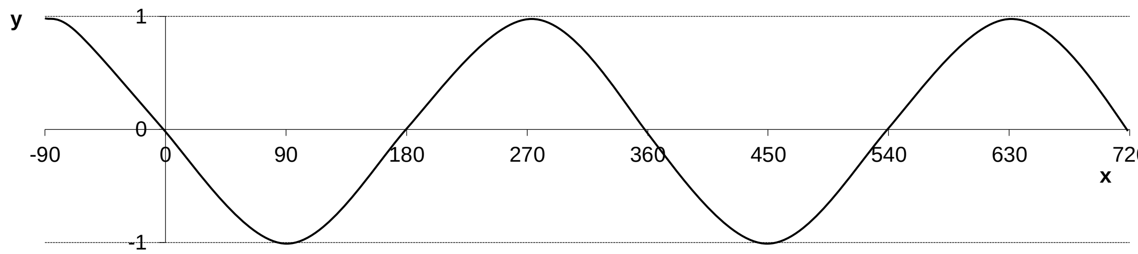


For the function above, $y = \cos 2x$ can be a more convenient formula than $y = \sin 2(x + 45)$

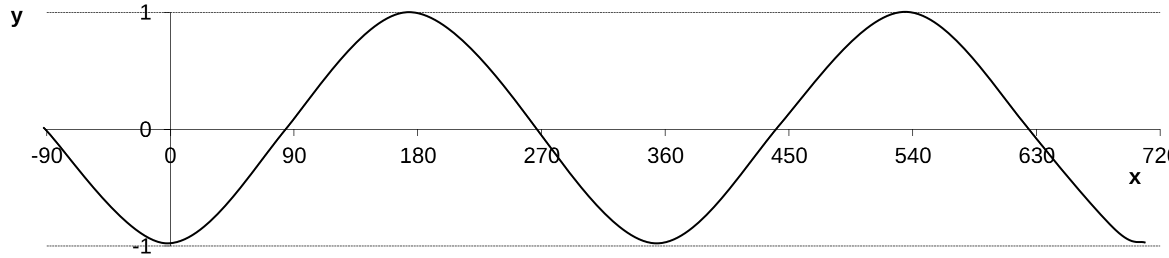
We often write sinusoidal functions in terms of cosines if it means that there will not be a phase shift or there will be just a small phase shift.

In the same way, we sometimes write sinusoidal functions as negative sine functions or negative cosine functions like this:

$y = -\sin x$



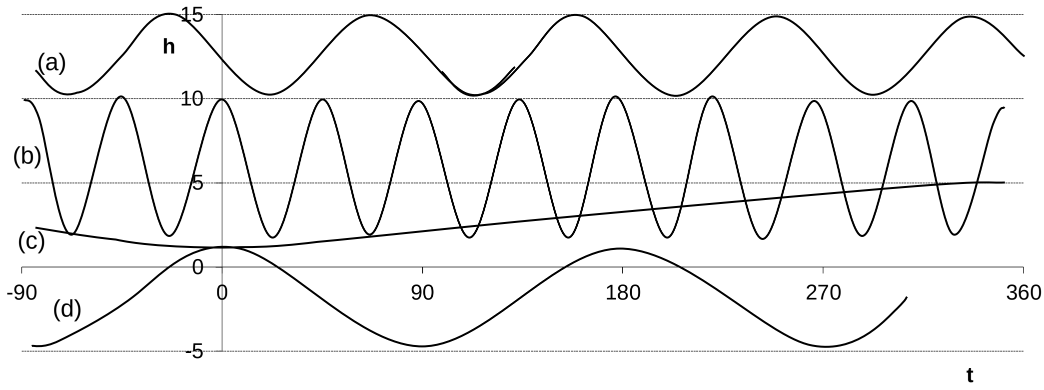
$y = -\cos x$



Having a choice of $\sin x$, $-\sin x$, $\cos x$ and $-\cos x$ makes it easier to avoid large phase shifts when writing formulae.

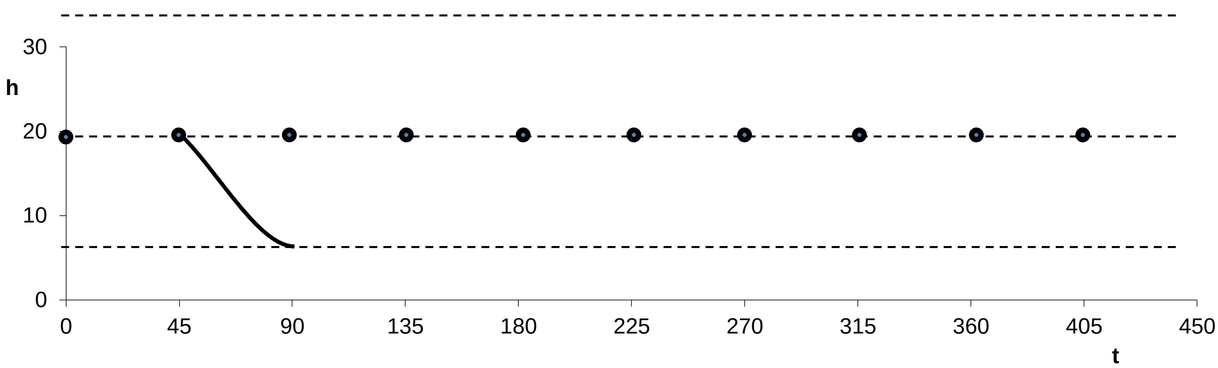
Practice

Q4 Write formulae with $c = 0$ for each of the following functions.

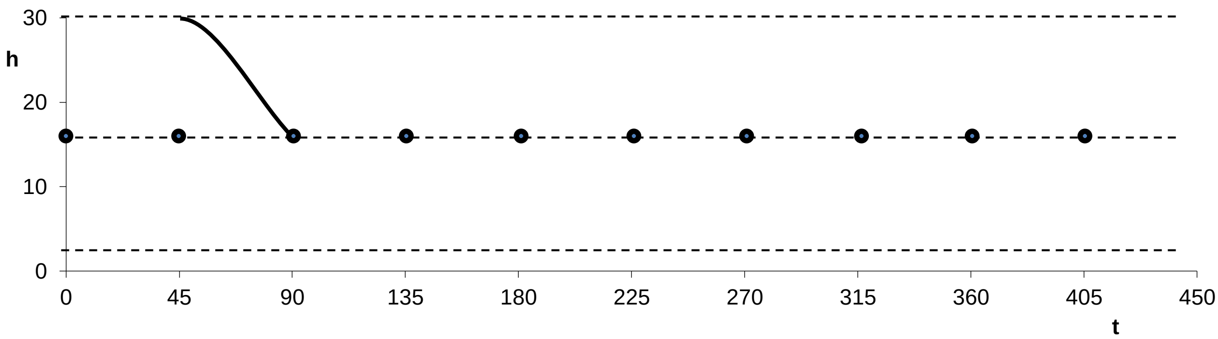


When drawing a graph involving a cos or a negative sin or cos, we follow the same procedure as above, except that, when we draw the first quarter cycle, we don't start at the middle and go up; we draw it the same way that the −sin t, cos t or −cos t goes.

For instance, if we were graphing $h = -14 \sin 2(t - 45) + 16$, we would start at the mean position and go down to the bottom



If we were graphing $h = 14 \cos 2(t - 45) + 16$, we would start at the top and go down to the mean position



for −cos we start at the bottom and go up to the mean position

Practice

Q5 Without your calculator, sketch graphs of the following sinusoidal functions. Then use your calculator to check your sketch.

(a) $y = \cos x$

(b) $y = -\sin (x - 90) + 4$

(c) $y = -5 \cos x$

(d) $y = 4 \cos 2(x + 15) + 1$

(e) $y = -\frac{1}{2} \sin 4(x + 5) + 20$

(f) $y = -5 \cos \frac{1}{2}(x - 180)$

Using Transformations

The relations between the parameters and the characteristics of sinusoidal functions can be explained using the ideas from the algebraic transformations module (A6-2). If you have done this module, see if you can do this.

Life-Related Problems

Here are some life-related problems which use the ideas in this module.

Practice

Q6 There is a high tide of 2.7 m at 12:36 p.m., then a low tide of 0.9 m at 6:36 p.m. Assuming that tide height is a sinusoidal function of time, write a formula for tide height in terms of time since midday in hours.

Q7 The average midday temperature in Birmingham is given by the formula $T = 6 \cos 30(t - 1) + 14$, where T is the temperature in °C and t is the time in months since the start of the year. Sketch this relation as a graph and use it to estimate the average midday temperature on October 1.

Q8 A 100 g weight is hanging on the end of a spring. At rest it is 100 cm above the ground. It is pulled down a distance of 10 cm and released. It then rises and falls sinusoidally with an amplitude of 10 cm and a period of 2 s. Sketch a graph of the relation between height above the floor in centimetres and time since release in seconds. Then express this relation as a formula.

Q9 The voltage of the live mains wire varies with time according to the formula $V = 340 \sin 18000t$, where V is the voltage in volts and t is the time in seconds. Find the amplitude and frequency of the voltage variations.

Q10 A Ferris wheel takes 2 minutes to do a revolution. The radius is 8 m and the axel is 10 m above the ground. If a car starts at the lowest point at

$t = 0$, find a formula for the height in terms of time since starting. Roughly how high will the car be 5 min 40 s after starting?

Q11 A buoy is floating on the top of the sea. Because of the swell, the distance, h, between the buoy and the sea bed varies with time, t, according to the formula $h = 0.6 \sin 60t + 7$, where h is in metres and t is in seconds. How many waves pass per minute? Over what range of heights does the buoy travel?

Q12 A weight on a string is swinging very gently. At time t (in seconds) the weight is d cm to the right of its equilibrium position. If the pendulum swings 40 times per minute and the maximum distance from the equilibrium position is 4 cm, write a formula for the relation between d and t. (Assume that the relation is sinusoidal.)

Non-sinusoidal Trigonometric Functions

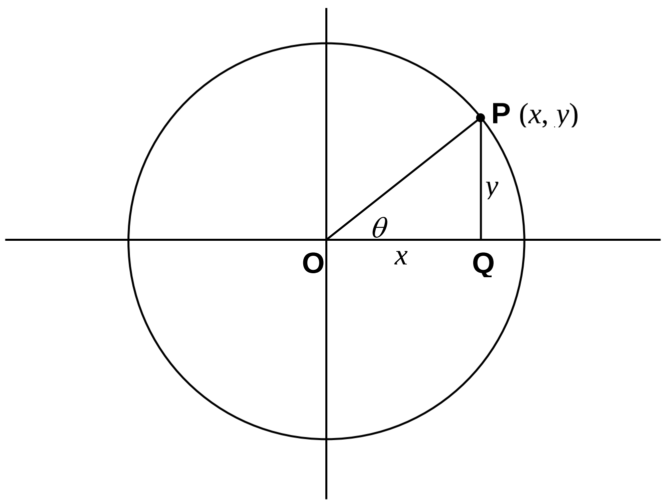
The functions above of the form $y = a \sin b(x + c) + d$ or $y = a \cos b(x + c) + d$ are sinusoidal functions. The family of sinusoidal functions is a sub-family of the family of trigonometric functions. Trigonometric functions are any functions involving trigonometric ratio – sin, cos or tan.

The following are some examples of trigonometric functions which are not sinusoidal functions:

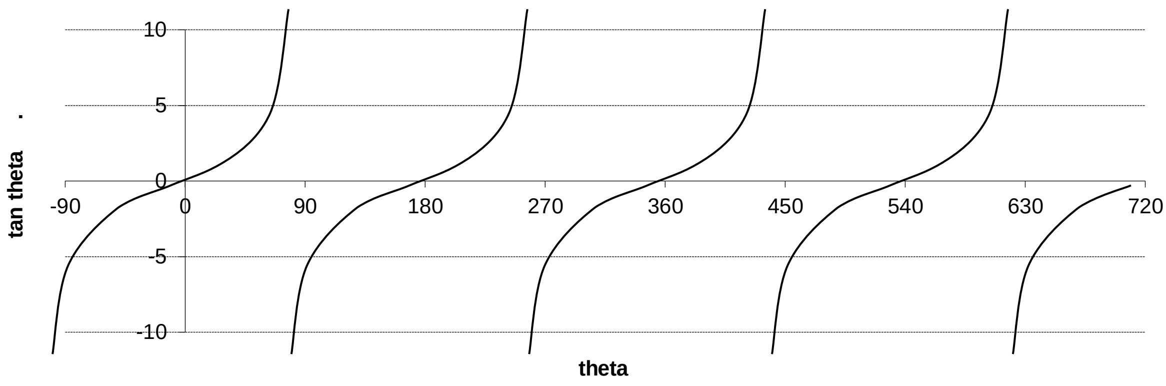
$y = \tan x$ $y = 3 \tan 2x$ $y = \sin^2 x$ $y = \frac{1}{\sin x}$ $y = \sin x + \sin 2x$

Non-sinusoidal functions are used a lot less than sinusoidal functions. All you really need to know about them here is what the graph of $y = \tan x$ looks like.

You should remember that $\tan \theta = \frac{\sin \theta}{\cos \theta}$. This is $\frac{y}{x}$ on the circle diagram to the right and $\frac{y}{x}$ is the gradient of OP. So $\tan \theta$ is given by the gradient of OP



By considering the gradient of OP as θ increases, we get the following graph for $y = \tan \theta$.



Note that at $\theta = 90°$ and every 180° thereafter, OP becomes vertical and its gradient infinite or undefined.

Practice

Q13 Use the graph of $y = \tan\,\theta$ and your knowledge of transformations or the ideas from above to sketch the following functions. Then use your calculator to check your sketch.

(a) $y = \tan\,\theta + 2$ (b) $y = 0.1\tan\,\theta$

(c) $y = \tan\,2\theta$ (d) $y = 2\tan\,(\theta + 45) - 1$

Solve

Q51 A point is moving in a clockwise circle centred on the origin of the Cartesian plane such that its y-coordinate is given by $y = 4\sin 60\,t$, where t is time in seconds.

(a) What is the formula for its x-coordinate?

(b) What are its coordinates when $t = 8$?

Q52 A point moves on the Cartesian plane such that, at time t, its x-coordinate is $\sin t$ and its y-coordinate is $\sin 2t$. What shape does it trace out?

Q53 A point moves on the Cartesian plane such that the relation between its distance from the origin, r, and its angle, θ, anti-clockwise from positive x-axis is $r = \sin 5\theta$. What shape does it race out?

Q54 From your knowledge of what the graph of $y = \cos x$ looks like, sketch the following functions. Check your sketches with a calculator.

(a) $y = \cos^{-1} x$ (b) $y = \frac{1}{\cos x}$ (c) $y = \cos^2 x$

Revise

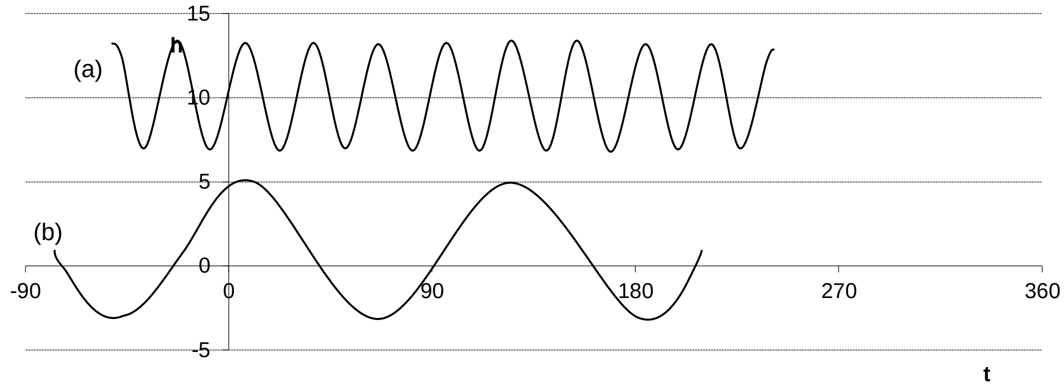
Revision Set 1

Q61 For each of these functions,

(iv) give the mean position, the amplitude, the period and the phase shift,

(v) then find the values of the parameters a, b, c and d,

(vi) then find the formula.



Q62 Without your calculator, sketch graphs of the following functions. Then use your calculator to check your sketch.

(a) $h = 2 \sin 6(t + 20) + 4$ (c) $h = -5 \cos 2(t - 30) - 4$

Q63 The latitude at which the sun is overhead at midday varied sinusoidally between 23.5° on June 21 and −23.5° on December 21. If time is measured in months since January 1, find the amplitude, period, phase shift and mean position of the relation between latitude and time, then write the relation as a formula in terms of cos.

Answers

Q1 (a) (i) mp = 3, amp = 4, ps = 0, p = 180 (ii) $a = 4$, $b = 2$, $c = 0$, $d = 3$
(iii) $h = 4 \sin 2t + 3$
(b) (i) mp = 7, amp = 2, ps = −20, p = 180 (ii) $a = 2$, $b = 2$, $c = 20$, $d = 7$
(iii) $h = 2 \sin 2 (t + 20) + 7$
(c) (i) mp = 15, amp = 3, ps = −10, p = 45 (ii) $a = 3$, $b = 8$, $c = 10$, $d = 15$
(iii) $h = 3 \sin 8 (t + 10) + 3$
(d) (i) mp = −3, amp = 2, ps = −90, p = 360 (ii) $a = 2$, $b = 1$, $c = 90$, $d = -3$
(iii) $h = 2 \sin (t + 90) - 3$

Q2 (a) (i) mp = −1, amp = 3, ps = 0, p = 180 (ii) $a = 3$, $b = 2$, $c = 0$, $d = -1$
(iii) $h = 3 \sin 2t + 7$
(b) (i) mp = 3, amp = 2, ps = −180, p = 720 (ii) $a = 2$, $b = 0.5$, $c = 180$, $d = 3$
(iii) $h = 2 \sin 0.5 (t + 180) + 3$
(c) (i) mp = 5, amp = 3, ps = −10, p = 45 (ii) $a = 3$, $b = 8$, $c = 10$, $d = 5$
(iii) $h = 3 \sin 8 (t + 10) + 5$
(d) (i) mp = 12.5, amp = 2.5, ps = 40, p = 90 (ii) $a = 2.5$, $b = 4$, $c = -40$, $d = 12.5$

(iii) $h = 2.5 \sin 4(t - 40) + 12.5$

Q4 (a) $h = -2.5 \sin 4t + 12.5$
(b) $h = 4 \cos 8t + 6$
(c) $h = -2 \cos 0.5t + 8$
(d) $h = 3 \cos 2t - 2$

Q6 $h = 4 \cos 15(t - 0.6) + 1.8$

Q7 Check the graph with your calculator, 11°

Q8 $h = -10 \cos 180(t + 10) + 100$, Check the graph with your calculator

Q9 amplitude = 340, frequency = 50Hz

Q10 $h = -8 \cos 3t + 10$, where t is in seconds, 6 m

Q11 10, 6.4 m to 7.6 m

Q12 $d = 4 \sin 240t$ (assuming it is at the equilibrium position and moving in the positive direction when t = 0)

Q51 (a) $y = -4 \cos 60t$ (b) (3.46, 0.14)

Q52 ∞

Q53 A flower with 5 petals

Q61 (a) (i) mp = 10, amp = 3, ps = 0, p = 30 (ii) $a = 3$, $b = 12$, $c = 0$, $d = 10$
(iii) $h = 3 \sin 12t + 10$
(b) (i) mp = 1, amp = 4, ps = 10, p = 120 (ii) $a = 4$, $b = 3$, $c = -10$, $d = 1$
(iii) $h = 4 \sin 3(t - 10) + 1$

Q63 amp = 47, p = 12, ps = 2.7, mp = 0, $latitude = 23.5 \cos 30(t - 2.7)$