M1 Maths

A5-10 Further Relations

- composition of functions
- inverse functions and $y^2 = x$
- discontinuities
- rational functions
- piecewise and step functions
- absolute value functions
- circles

Summary Learn Solve Revise Answers

Summary

Composition of Functions: If the operations of function f are performed on x, then the operations of g are performed on the result, the total process can be represented as $g(f(x))$ or $g \circ f(x)$.

Inverse Functions: The inverse function of a function $f(x)$ is the sequence of operations which will undo $f(x)$, producing x again. The inverse of $f(x)$ is written $f^{-1}(x)$. $f^{-1}(x)$ can be found by swapping $f(x)$ and x in the formula for $f(x)$, then changing f to f^{-1}, then rearranging to make f^{-1} the subject. The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$.

Discontinuities: A function $f(x)$ has a discontinuity at $x = a$ if it is not defined at $x = a$. Discontinuities can be asymptotes (where y increases to ∞ or decreases to $-\infty$ as x approaches a) or gaps (where y continues as normal up to and beyond a, but is not defined at a).

Rational Functions: A rational function is a function of the form $y = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. They are characterised by the presence of discontinuities (asymptotes or gaps) in their graphs.

Piecewise Functions: Piecewise functions are those functions which have different formulae for different parts of their domain. To define them, we list the formulae, each with the section of the domain over which it applies.

Step Functions: These are a particular type of piecewise function in which the value of the dependent variable rises (or falls) at certain regular values of the independent variable, but remains unchanged between those values. The graph of a step function is a set of horizontal line segments resembling the treads of a set of steps.

Absolute Value Functions: The absolute value of x, $|x|$, is equal to x if $x \geq 0$ and equal to $-x$ if $x < 0$. Functions which contain the $|\ |$ symbol are called absolute value functions.

Circles: The equation for a circle of radius r centred at (a, b) is $(x - a)^2 + (y - b)^2 = r^2$.

Learn

In this module you will learn a few miscellaneous facts about relations:

- composition of functions
- inverse functions and the relation $y^2 = x$
- discontinuities
- rational functions
- piecewise and step functions
- absolute value functions
- circles

Composition of Functions

Think of a function as something we do to a variable. For instance, the function $f(x) = 2x$ is the function 'multiply by 2'. $g(x) = \sin x$ is the function 'take the sine'. $h(x) = x^2 + 3x + 5$ is the function 'square, then add 3 times the original number, then add 5'.

We can start with x, then apply $f(x)$ to get x multiplied by 2, i.e. $2x$. Then we can apply $g(x)$ to that to get the sine of $2x$, i.e. $\sin 2x$.

$2x$ is a function of x. $\sin 2x$ is a function of $2x$, which is itself a function of x. So $\sin 2x$ is a function of a function of x. Using function notation, we can write it as $g(f(x))$.

Getting a function of a function is called composition of functions. $g(f(x))$ is a composite function. We have formed a composite of the functions f and g.

In this case, g is called the *outside function*. The outside function is the last one to be done. f is called the inside function. The *inside function* is the first to be done.

Of course we can then do something to $g(f(x))$ to get a function of a function of a function and so on. For example, $h(g(f(x))) = (\sin 2x)^2 + 3 \sin 2x + 5$.

The brackets make the expression $h(g(f(x)))$ a bit messy and hard to read, so we sometimes use a simpler notation. We write $h(g(f(x)))$ as $h \circ g \circ f(x)$. Both can be pronounced *h of g of f of x*, though the latter is sometimes pronounced *h o g o f of x*.

Examples

Suppose $m(x) = x + 1$, $n(x) = x^2$ and $p(x) = 10^x$.

(a) Find $n \circ m(x)$

We start by finding $m(x)$. [Note that we always start with the inside function (the letter closest to the (x) and work outwards.] $m(x) = x+1$. Then we find $n(x+1)$ which is $(x+1)^2$.

(b) Find $n \circ p \circ m(3)$

We start by finding $m(3)$, which is 3+1, i.e. 4. Then we find $p(4)$ which is 10^4, i.e. 10 000. Then we find $n(10\,000))$ which is $10\,000^2$, i.e.100 000 000.

Note that $n \circ p \circ m(x)$ is not generally equal to $p \circ n \circ m(x)$, $m \circ n \circ p(x)$ etc. Try them to see. The order of the functions matters.

Practice

Q1 If $f(x) = x + 3$, $g(x) = 2x^2$ and $h(x) = {}^1/_x$, find:

(a) $f(g(x))$	(b) $g(f(x))$	(c) $h(g(x))$
(d) $g \circ h(x)$	(e) $h \circ f(x)$	(f) $f \circ g(2)$
(g) $g \circ h(2)$	(h) $f \circ g \circ h(1)$	(i) $h \circ g \circ f(1)$
(j) $h \circ g(-2)$	(k) $f \circ h(a^2)$	(l) $g \circ h \circ f(-4n)$

Inverse Functions

Inverses of one-step functions

Suppose $f(x) = 5x$. This is the function that multiplies by 5.

Suppose $g(x) = x \div 5$. This is the function that divides by 5.

$f(x)$ and $g(x)$ will undo each other. For instance, start with 4, apply $f(x)$ and you get $4 \times 5 = 20$, then apply $g(x)$ and you get $20 \div 5 = 4$. We are back where we started.

We say that $f(x)$ and $g(x)$ are inverses of each other. We write the inverse of $f(x)$ as $f^{-1}(x)$.

The same goes for any other function. So if $h(x) = x - 3$, then $h^{-1}(x) = x + 3$.

Inverses of multi-step functions

Finding the inverse of a one-step function like $f(x) = 5x$ is straightforward – you just use the inverse operation. But how do we find the inverse of $f(x) = 3x + 5$?

One way is to decide what operations are used in applying the function. The inverse function will have the inverse operations applied in the reverse order. This is a very similar idea to solving equations by backtracking.

So if $f(x) = 3x + 5$, then $f^{-1}(x) = (x - 5) \div 3$.

Another way is to swap $f(x)$ and x and change $f(x)$ to $f^{-1}(x)$.

So $f(x) = 3x + 5$ becomes $x = 3f(x) + 5$, then $x = 3f^{-1}(x) + 5$.

Then we rearrange to make $f^{-1}(x)$ the subject:

$$x = 3f^{-1}(x) + 5$$

$$x - 5 = 3f^{-1}(x)$$

$$(x - 5) \div 3 = f^{-1}(x)$$

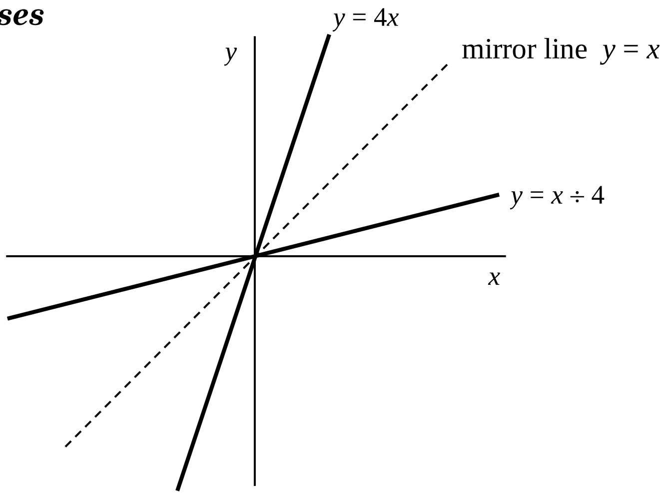
So $f^{-1}(x) = (x - 5) \div 3$ as we found before.

Every function has an inverse. The inverse will be a relation, though it may not always be a function. For instance the inverse of $y = x^2$ is $y = \pm\sqrt{x}$, which is not a function. But the inverse of $y = x + 4$ is $y = x - 4$, which is a function.

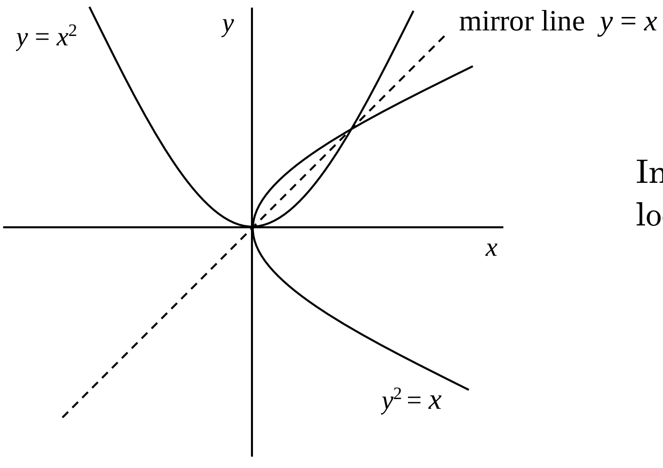
Graphs of functions and their inverses

If you graph a function and its inverse, they will be reflections of each other in the line $y = x$.

Take $y = 4x$ and $y = x \div 4$. They look like this:



In the same way $y = x^2$ and its inverse $y^2 = x$ look like this:



$y^2 = x$ is a parabola with a horizontal axis of symmetry and vertex at the origin.

Knowing that $y^2 = x$ is the inverse of $y = x^2$ gives us a quick way of knowing what the graph looks like: we know what $y = x^2$ looks like and $y^2 = x$ is its reflection in the line $y = x$.

If function f is the inverse of function g, then function g is also the inverse of function f. Each is a reflection of the other in the line $y = x$.

Practice

Q2 Find the inverse of each of the following functions:

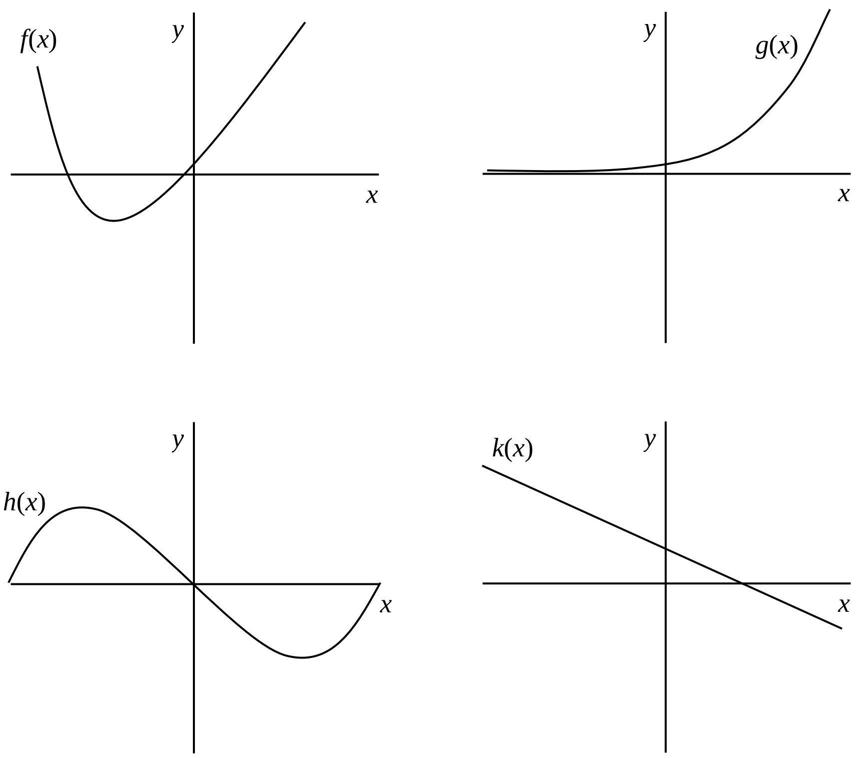
(a) $f(x) = x + 5$ (b) $g(x) = x \div 3$ (c) $h(x) = 3x + 7$

(d) $y(x) = \dfrac{3x - 4}{7}$ (e) $y = 4x$ (f) $r(t) = 3t - 5$

(g) $h = 4r^2$ (h) $k = 4a^3 + 6$ (i) $h = \sin t$

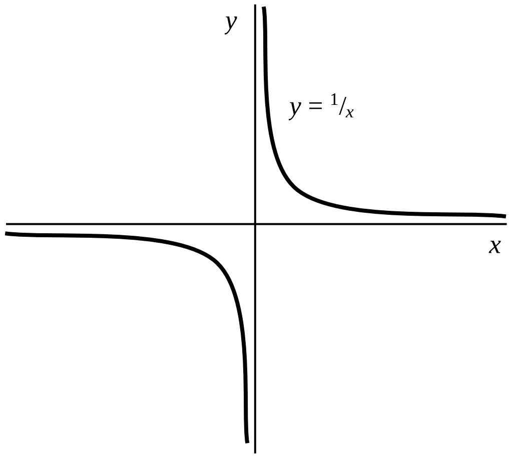
(j) $p = 2 \sin 3t + 20$ (k) $x = \cos^2 t$ (l) $y = \dfrac{3x^2 - 3}{5}$

Q3 Sketch the inverses of the functions shown below.



Discontinuities

Consider the function $y = \frac{1}{x}$. It is defined for all x-values except $x = 0$. If we graph it, we can theoretically graph it for all x-values except 0. The function is not defined at $x = 0$. We say that it has a discontinuity at $x = 0$. It is continuous for all other values of x, but it is discontinuous at $x = 0$.



As you can see from the graph, y approaches ∞ or $-\infty$ as x approaches 0.

Any function with the independent variable in the denominator will have a discontinuity when the denominator is 0.

We sometimes call a function with one or more discontinuities *a discontinuous function*. Discrete functions have discontinuities just about everywhere, so we don't bother to describe them as discontinuous.

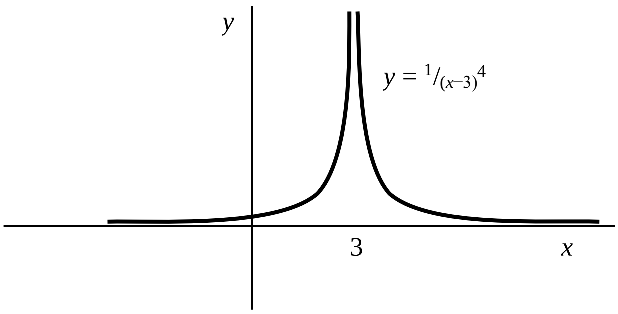
Ironically, therefore, only continuous functions can be discontinuous!

Mathematical language isn't always totally logical.

Asympyotes

The discontinuity in $y = \frac{1}{x}$ is of a type known as an asymptote. If a function has an asymptote at $x = a$, then, as x gets closer and closer to a, y increases to infinity or decreases to $-\infty$.

In most cases it increases as we approach the asymptote from one side and decreases as we approach it from the other side. An exception to this is if we have an even number of identical factors on the bottom of the fraction, e.g. $y = \frac{1}{x^2}$ or $y = \frac{1}{(x-3)^4}$. In such cases, the function increases or decreases from both sides of the asymptote. The graph to the right is of the function $y = \frac{1}{(x-3)^4}$.



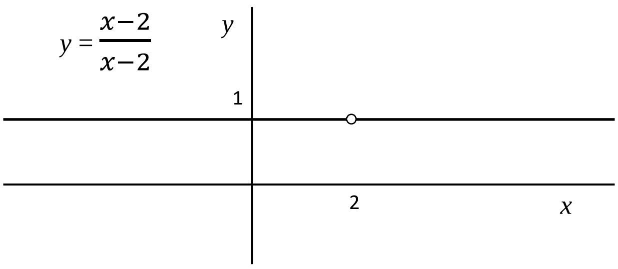
This function, $y = \frac{1}{(x-3)^4}$, has an asymptote at $x = 3$. But it also has an asymptote at $y = 0$ because x approaches ∞ or $-\infty$ as y approaches 0. We call the asymptote at $x = 3$ a vertical asymptote, the one at $y = 0$ a horizontal asymptote. (Note that a horizontal asymptote is not a discontinuity.)

As you can see, asymptotes may have value pairs on both sides (as in the asymptote at $x = 3$ in $y = \frac{1}{(x-3)^4}$) or just on one side (as in the asymptote at $y = 0$).

Make sure you can spell *asymptote*.

Gaps

A different type of discontinuity occurs in the function $y = \frac{x-2}{x-2}$. In this function, $y = 1$ for all x-values except $x = 2$. There is an infinitely small gap in the graph at $x = 2$. We show it like this:



Discontinuities are not of much practical significance and any problems that arise as a result of discontinuites can generally be worked out with a bit of common sense. Knowledge of asymptotes is, however, useful in sketching graphs of rational functions, which you might do later.

Practice

Q4 For what values of x (if any) are the following functions discontinuous? Specify the type of each discontinuity (asymptote or gap).

(a) $y = \frac{1}{x-3}$ (b) $y = \frac{1}{x^2}$ (c) $y = (x + 1)^2$

(d) $A = \frac{r^3}{(r+1)^2}$ (e) $y = \frac{x-3}{x-3}$ (f) $y = \frac{x-1}{x(x-1)}$

Rational Functions

A rational function is a function of the form $y = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Their distinguishing feature is the presence of discontinuities (asymptotes or gaps). If $P(x)$ and $Q(x)$ are expressed as products of linear factors, then there is a discontinuity at every x-value where one of the factors of $Q(x)$ equals zero. If there is an identical factor in $P(x)$, the discontinuity will be a gap; if not, it will be an asymptote.

For example, $y = \frac{5(x-1)(x+2)}{(x-1)(x+3)}$ will have a gap at $x = 1$ and a vertical asymptote at $x = -3$.

Graph it on your calculator to have a look, but note that your calculator will not show the gap.

Because the top and bottom of the fraction are polynomials of the same degree (degree 2), this means that as x becomes very large, y will approach a particular value, in this case 5. So there is a horizontal asymptote at 5 also.

If the bottom is of higher degree than the top, then the horizontal asymptote will be at $y = 0$; if the top is of higher degree, then the graph will continue to increase or decrease as x becomes very large, positive or negative, so there won't be a horizontal asymptote.

Practice

Q5 Give the formula for a rational function which fits each of the following descriptions. Graph your formula on your calculator to check.

(a) asymptotes at $y = 0$ and $x = 3$

(b) gap at $x = -2$, y increasing to infinity as x becomes very large positive and very large negative

(c) asymptotes at $x = 1$, $x = 3$ and $y = 5$

Piecewise Functions

Consider a fairground ride in which one is carried up from ground level to 12 m at 2 m/s, then held there for 3 s, then dropped back onto the ground. If h is the height above the ground in metres and t is the time in seconds since the leaving the ground, then the relation between h and t is:

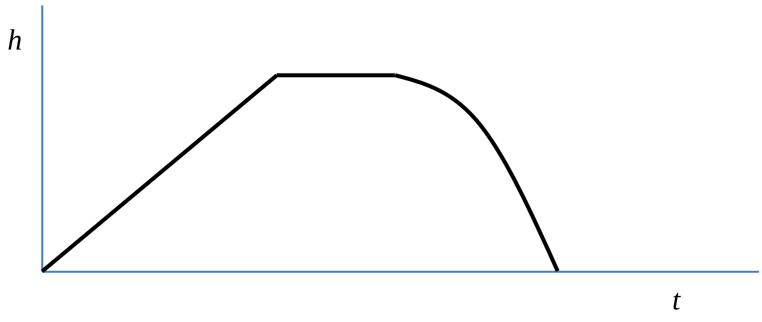
$h = 2t$ for $0 \leq t \leq 6$

$h = 12$ for $6 < t \leq 9$

$h = 12 - 4.9(t - 9)^2$ for $9 < t \leq 10.565$

The function has three different formulae for different parts of the domain. A function like this is called a piecewise function.

Its graph would have three parts like this:



Practice

Q6 Write a piecewise function to represent the relation between the cost to post a parcel and its mass if a parcel under 0.5 kg costs \$7.50, one from 0.5 to 3 kg costs \$15/kg and one over 3 kg costs \$45 plus \$10/kg for the mass in excess of 3 kg.

Step Functions

A common type of piecewise function is the step function. The cost of posting a parcel is actually more likely to be something like this: \$9 plus \$2 for each kilogram or part thereof over 2 kg.

If c is the cost in dollars and m the mass in kilograms, then, as a formula, this function would be:

$c = 9$ for $0 < m \leq 2$

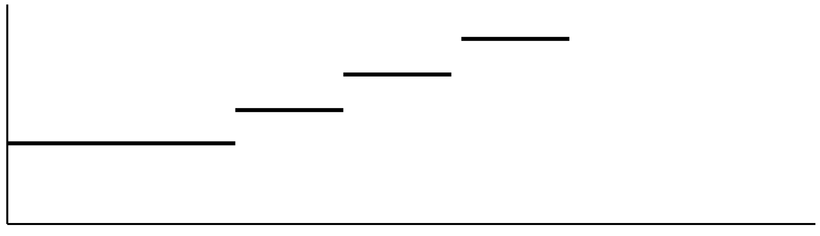
$c = 11$ for $2 < m \leq 3$

$c = 13$ for $3 < m \leq 4$

$c = 15$ for $4 < m \leq 5$

and so on.

As a graph, this would be something like this:



Practice

Q7 Write a step function to represent the relation between the cost and time spent on a job if a job of 1 hour or less costs \$120 and each extra half hour or part thereof costs a further \$30.

Absolute Value Functions

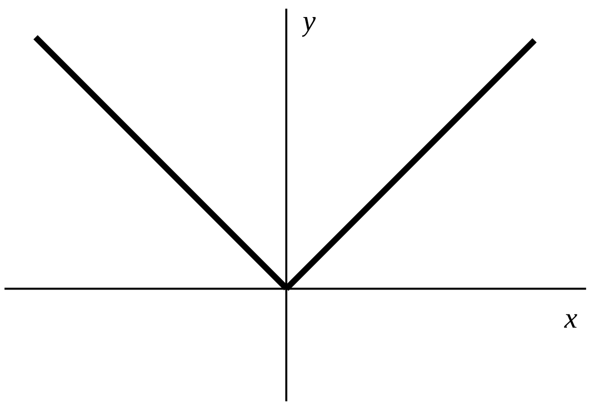
One other type of function worth knowing about is the absolute value function.

The absolute value of a number is the same number, but with any negative sign removed. So the absolute value of –5 is 5; the absolute value of 7 is 7; the absolute value of 0 is 0; the absolute value of –1.73 is 1.73.

The absolute value of a number is written symbolically by putting vertical lines either side of the number like this: $|2|$, $|-4|$, $|-2.27|$, $|x|$, $|-a^3|$.

So we can define absolute value by: $|x| = x$ if $x \geq 0$; $|x| = -x$ if $x < 0$.

The graph of $y = |x|$ looks like this:



Practice

Q8 Use the graph of $y = |x|$ and your knowledge of transformations to sketch the following absolute value functions. Check your sketches with your calculator.

(a) $y = 5|x|$ (b) $y = -2|x|$

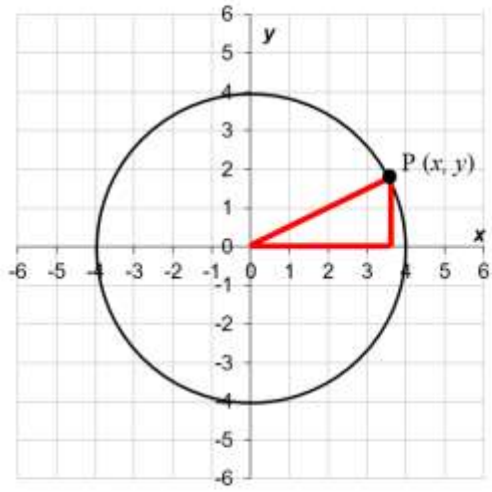
(c) $y = |5x|$ (d) $y = |-2x|$

(e) $y = |x| + 3$ (f) $y = |3 - x|$

(g) $y = 3|x + 1|$ (h) $y = |4 - 2x|$

(i) $y = \frac{1}{2}|3x + 2|$ (j) $y = 4|3x - 2| + 1$

Circles



The formula for a circle of radius r centred on the origin is $x^2 + y^2 = r^2$. This should be obvious by applying Pythagoras to this diagram.

So the circle in the diagram is $x^2 + y^2 = 16$.

Circles in different sizes can be obtained by changing the value of r. For instance $x^2 + y^2 = 3$ will have a radius of $\sqrt{3}$.

Circles in other locations can be obtained by applying translations to the formula. For instance $(x - 3)^2 + (y + 1)^2 = 25$ is a circle or radius 5 centred at $(3, -1)$.

Practice

Q9 Without using your calculator, draw graphs of the following relations. Draw them all on the same set of axes, but label them. Use your calculator to check your answers. (You might need to make y the subject of the formula before you graph it.)

(a) $x^2 + y^2 = 4$ (b) $(x + 3)^2 + y^2 = 4$

(c) $x^2 + (y - 4)^2 = 4$ (d) $(x + 1)^2 + (y - 4)^2 = 9$

(e) $(3x)^2 + y^2 = 9$ (f) $2(x + 1)^2 + ½(y - 2)^2 = 16$

Q10 Find the equation of a circle with radius 4 centred at $(5, -2)$.

Solve

Q51 If $f(x) = \ln(\sin x^2 - 3)$, what is $f(f^{-1}(x))$?

Q52 List the discontinuities in $y = \frac{x^2-3x+2}{x^3-5x^2+6x}$

Q53 Posting a parcel costs \$12 for up to 1 kg plus \$4 for every extra kg or part thereof. Parcels over 6 kg cannot be posted. Assuming that parcels are equally likely to be any mass from 0 to 6 kg, what is the average parcel postage cost?

Q54 Look at the graph of $y = x^3 - x$ on your calculator. Then sketch the graphs of $y = |x^3 - x|$ and $y = |x^3| - |x|$. Are they the same? Check your sketches on your calculator.

Q55 Find the formula for an ellipse with domain $2 \le x \le 6$ and range $-7 \le y \le 1$.

Revise

Revision Set 1

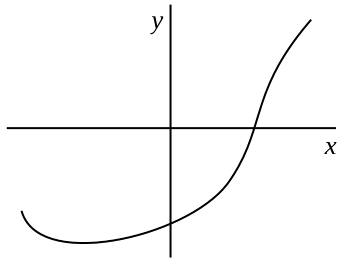
Q61 If $f(x) = x + 1$, $g(x) = 5x^2$ and $h(x) = \sin x$, find:

(a) $f(g(x))$ (b) $g \circ h \circ f(a)$

Q62 Find the inverse of each of the following functions:

(a) $f(x) = 3x - 5$ (b) $y(x) = \frac{2x^3}{5}$

Q63 Sketch the inverse of the function to the right.



Q64 For what values of x (if any) are the following functions discontinuous?

(a) $y = \frac{4x+2}{x(x-2)}$

Q65 Write a piecewise function to represent the relation between the cost to post a parcel and its mass if a parcel under 0.5 kg costs \$11, one from 0.5 to 2 kg costs \$15 and one over 2 kg costs \$30 plus \$2/kg for the mass in excess of 2 kg.

Q66 Sketch the function $y = 5|2 - x|$. Check your sketch with your calculator.

Q67 Sketch the graph of $x^2 + (y - 4)^2 = 25$.

Answers

Q1 (a) $2x^2 + 3$ (b) $2(x + 3)^2$ (c) $\frac{1}{2x^2}$

(d) $\frac{2}{x^2}$ (e) $\frac{1}{x+3}$ (f) 11

(g) ½ (h) 5 (i) $^1/_{32}$

(j) $^1/_8$ (k) $\frac{1}{a^2} + 3$ (l) $\frac{2}{(3-4n)^2}$

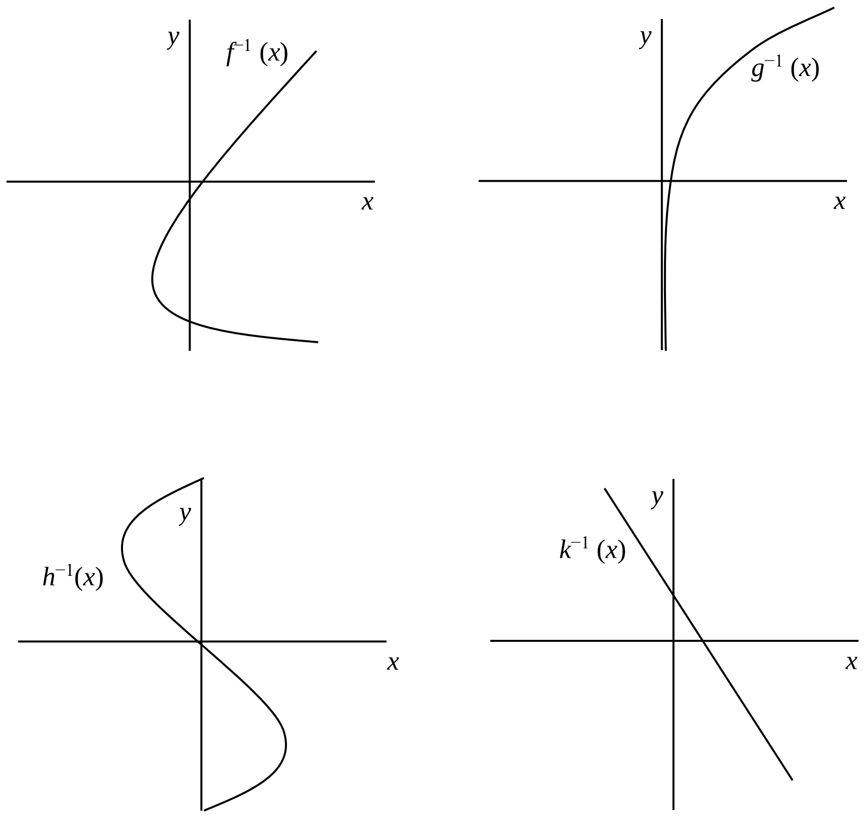
Q2 (a) $f^{-1}(x) = x - 5$ (b) $g^{-1}(x) = x \times 3$ (c) $h^{-1}(x) = \frac{x-7}{3}$

(d) $y^{-1}(x) = \frac{7x+4}{3}$ (e) $y^{-1} = \frac{x}{4}$ (f) $r^{-1}(t) = \frac{t+5}{3}$

(g) $h^{-1} = \pm\sqrt{\frac{r}{4}}$ (h) $k^{-1} = \sqrt[3]{\frac{a-6}{4}}$ (i) $h^{-1} = \sin^{-1} t$

(j) $p^{-1} = {}^1/_3 \sin^{-1}\left(\frac{t-20}{2}\right)$ (k) $x^{-1} = \pm \cos^{-1} \sqrt{t}$ (l) $y^{-1} = \pm\sqrt{\frac{5x+3}{3}}$

Q3



Q4 (a) 3 (asymptote) (b) 0 (asymptote) (c) none
(d) −1 (asymptote) (e) 3 (gap) (f) 0 (asymptote), 1 (gap)

Q5

Q6 $c = 7.5$ for $0 < m < 0.5$
$c = 15m$ for $0.5 \le m \le 3$
$c = 45 + 10m$ for $m \ge 3$, where c is the cost in dollars and m is the mass in kilograms

Q7 $c = 120$ for $t \le 1$
$c = 150$ for $1 < t \le 1.5$
$c = 180$ for $1.5 < t \le 2$
etc.

Q10 $(x - 5)^2 + (y + 2)^2 = 16$

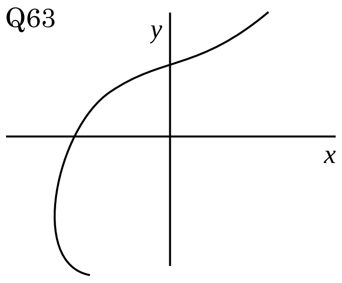
Q51 x Q52 0, 2, 3 Q53 $22 Q54 Not the same

Q55 $2(x - 4)^2 + (y + 3)^2 = 16$

Q61 (a) $5x^2 + 1$ (b) $5\sin^2(a + 1)$

Q62 (a) $f^{-1}(x) = \frac{x+5}{3}$ (b) $y^{-1}(x) = \sqrt[3]{2.5x}$

Q63



Q64 0, 2

Q65 $c = 11$ for $0 < m < 0.5$

$c = 15$ for $0.5 \le m \le 2$

$c = 30 + 2m$ for $m \ge 2$, where c is the cost in dollars and m is the mass in kilograms

Q67 Circle of radius 5, centred at (0, 4).