M1 Maths

A5-1 Polynomial Functions

- general form and graph shape
- equation solution methods
- applications

Summary Learn Solve Revise Answers

Summary

A polynomial function is made of a number of terms. Each term consists of a whole-number power of the independent variable multiplied by a real number.

The graph of a polynomial consists of a number of straighter sections called arms separated by more curved sections called elbows. The number of arms is less than or equal to the degree of the polynomial (the highest power of the independent variable).

You already know how to solve polynomial equations up to degree 2. In general, higher-degree equations can only be readily solved by graphing.

Linear and quadratic functions are the most commonly used polynomials, but higher-degree polynomials have applications in volume, trigonometry etc.

Learn

General Form

You know a bit about linear and quadratic functions. But these are part of a larger family called polynomial functions.

- Constant functions are of the form $y = c$
- Linear functions are of the form $y = mx + c$
- Quadratic functions are of the form $y = ax^2 + bx + c$

This sequence can be continued . . .

- Cubic functions are of the form $y = ax^3 + bx^2 + cx + d$
- Quartic functions are of the form $y = ax^4 + bx^3 + cx^2 + dx + e$

You will probably notice a pattern here.

The pattern continues . . .

- Quintic functions are of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$
- Sexual functions are of the form $y = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

[Of course *sexual function* has a completely different meaning in common English – be careful not to confuse the two.]

- Septic functions are of the form $y = ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h$

And so on *ad infinitum* . . .

Functions of all these types are collectively known as *polynomial functions*. *Polynomial* comes from the Greek for many numbers. It has nothing to do with hungry parrots.

A polynomial function is made of a number of terms. Each term consists of a whole-number power of the independent variable multiplied by a real number.

4 can be a term in a polynomial because it is $4 \times x^0$.

$2.7x$ can be a term in a polynomial because it is $2.7 \times x^1$.

x^2 can be a term in a polynomial because it is $1 \times x^2$.

$-0.4x^5$ can be a term in a polynomial because it is $-0.4 \times x^5$.

$\frac{1}{x}$, $\frac{3}{x^2}$, $6\sqrt{x}$ and $\sqrt[3]{x^4}$ cannot be terms in a polynomial because the powers of x are not whole numbers. Likewise 2^x, $\sin x$ etc. cannot be terms in a polynomial. Any function containing such a term is not a polynomial.

The degree of a polynomial is the highest power of the independent variable. So constant functions are polynomials of degree 0; linear functions are polynomials of degree 1; quadratics are polynomials of degree 2 and so on.

Practice

Q1 For each of the following functions, say whether it is a polynomial. If it isn't, say why. If it is, give its degree.

(a) $y = 3x - x^3$ (b) $y = 2.9 + 4x^2 - x^7$

(c) $h = 4t - 1$ (d) $g = \frac{2}{f} + 3f$

(e) $r = 6$ (f) $y = \sqrt{2}x + 3x^2$

(g) $g = x^{5.5}$ (h) $y^2 = 2 + 5x - x^2$

(i) $y = x^{12}$ (j) $k = 0$

(k) $y = (x - 3)(x + 4)$ (l) $h = 7.2(2x - 3)(x^2 - 5x + 2)$

Shapes of Polynomial Graphs

Practice

Q2 Graph the following polynomial functions on your graphics calculator using a window of $-10 \le x \le 10$, $-20 \le y \le 20$, then copy them.

(a) $y = 3x - 2$

(b) $y = 3x^2 - x$

(c) $y = -x^3 + 4x^2 + 5x - 6$

(d) $h = t^3 + t$

(e) $g = a^4$

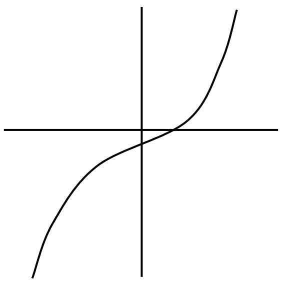
(f) $r = 0.2x^4 - 0.8x^3 - 2x^2 + 3x + 7$

(g) $g = 0.1x^5 - 4x^2 + 2$

(h) $y = x^6 - 3x^2 + 5x + 2$

Graphs of polynomials vary in shape, but all have certain features in common.

In quadratics and higher degree polynomials, as you move a long way to the left or right, the graph goes up towards ∞ or down towards $-\infty$ as in the example to the right



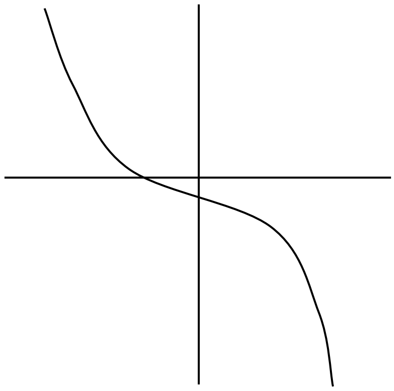
To decide which way it goes, just sub a large positive number for x (e.g. 1000) and see whether y is positive or negative. Then do the same with a large negative number (e.g. –1000).

This of course will depend on the sign of the coefficient of the largest power of x. Use this idea to complete the table below.

degree	coefficient of highest power of x	up or down to the left	up or down to the right
2	positive (e.g. $3x^2$)	up	up
2	negative (e.g. $-½x^2$)	down	down
3	positive (e.g. x^3)	down	up
3	negative (e.g. $-2x^3$)		
4	positive (e.g. $12x^4$)		
4	negative (e.g. $-x^4$)		
5	positive (e.g. $0.01x^5$)		

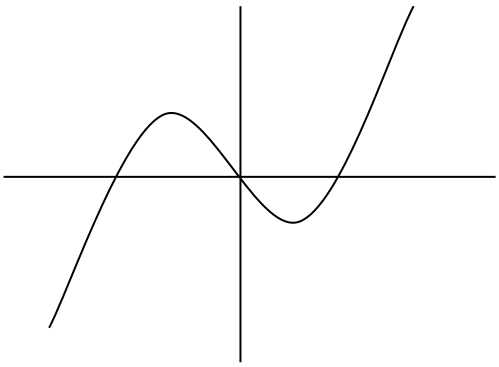
5	negative (e.g. $-0.2x^5$)		
6	positive (e.g. $6x^6$)		
6	negative (e.g. $-4x^6$)		

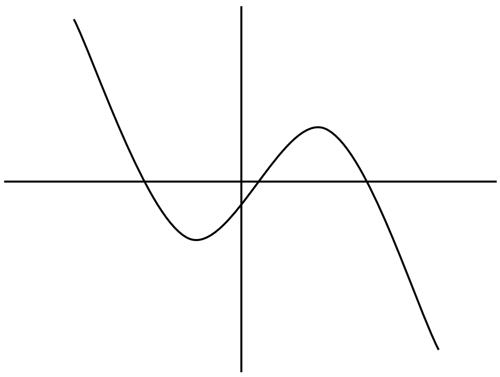
The graph will consist of a number of straighter sections called arms separated by more curved sections called elbows – like in the picture to the right.



A polynomial of degree n can have up to n arms and n–1 elbows, though it may have less.

If the graph changes from positive to negative gradient (or vice versa) across an elbow, then the elbow will be a turning point as in the graphs below. [A turning point is a point where the graph changes from a positive gradient through a zero gradient (horizontal) to a negative gradient or from a negative gradient through a zero gradient to a positive gradient.]





It is difficult to sketch the graph of a cubic or higher order polynomial without a graphics calculator, but the ideas above will give the likely general shape and the constant term will give the y-intercept. For example, $y = 3x^3 - 2x^2 + 7x - 4$ has a y-intercept of –4. If you do calculus later on, you will learn techniques for finding the locations of the turning points.

Practice

Q3 Use a graphing calculator to sketch the following polynomial functions. Show scales on the axes.

(a) $y = x^3 - 3x + 2$

(b) $y = x^4 - 4x^2 + 2x$

(c) $h = 4t^5 - t^4 + 2t^3 + t^2 - 3$

(d) $g = 3f^2 + 4f - 2$

Q4 Use a graphing calculator and a bit of guess and check to find a quartic function which crosses the x-axis four times

Q5 What is the largest number of turning points that a sexual function (degree 6) could have? What is the smallest number?

Q6 What is the largest number of turning points that a cubic polynomial (degree 3) could have? What is the smallest number?

Solving Equations from Polynomial Functions

You already know how to solve polynomial equations up to degree 2, i.e. linear and quadratic functions. Equations from polynomials of degree greater than 2, like $x^3 + 2x^2 + 8x - 3 = 0$ are hard to solve algebraically and we generally solve them by graphing. A polynomial equation of degree n can have up to n solutions.

A polynomial equation expressed with zero on the right of the = sign can be solved by factorising if the solutions are small integers.

$x^3 - 2x^2 - x - 2 = 0$ can be factorised into $(x + 1)(x - 1)(x - 2) = 0$, which can then be solved by the null factor theorem to give $x = -1$ or $x = 1$ or $x = 2$.

We do the factorisation by guess and check. Basically we take a guess at a solution to the equation. We might guess that $x = 1$ is a solution. If it is, then substituting 1 for x will make the equation true. Doing so gives

$$1^3 - 2 \times 1^2 - 1 - 2 = 0$$

This is true, which means that $x = 1$ is a solution and that $(x - 1)$ is a factor.

This is an application of the *Factor Theorem* which states that if substituting $x = a$ into a polynomial equation in the form $f(x) = 0$ makes it true, then $(x - a)$ is a factor of $f(x)$.

Having found a factor, it is then possible to divide $f(x)$ by that factor to get a polynomial of lower degree. You can then repeat this process until the remaining polynomial is a quadratic which you can factorise the way you learnt in Module A4-1.

The process of dividing the polynomial by a linear factor is called the **polynomial division algorithm**. If you need to know how to use this algorithm, the working in the box below explains it. It is quite similar to the long division algorithm for numbers. It could be worth reminding yourself of that by doing one before reading on.

Solving $x^3 + 3x + 10 = 6x^2$ by factorising

Put the equation into standard form

$x^3 - 6x^2 + 3x + 10 = 0$

Use guess and check (the factor theorem) to find one factor.

Try $x = 1$: $1^3 - 6\times1^2 + 3\times1 + 10 = 8$ No

Try $x = 2$: $2^3 - 6\times2^2 + 3\times2 + 10 = 0$ Yes

Divide by that factor, $x - 2$

So $x - 2$ is a factor

1. Divide the first term of the polynomial (x^3) by the first term of the factor (x) to get x^2. Write this above the x^2 term of the polynomial.
2. Then multiply the x^2 by the $x - 2$ and write the product, $x^3 - 2x^2$ below the appropriate terms of the polynomial.
3. Then subtract the $x^3 - 2x^2$ from the $x^3 - 6x^2$ above it to get $-4x^2$ and write this in the x^2 column.
4. Then bring down the next term of the polynomial – the $+ 3x$.

Then repeat steps 1 to 4 this time dividing the first term of $-4x^2 + 3x$ by the first term of the factor . . .

The fact that the final remainder is 0 confirms the fact that $x - 2$ divides evenly into $x^3 - 6x^2 + 3x + 10$ and so is a factor.

Finally, factorise the quadratic and solve.

$$
\begin{array}{r|rrrrrrr}
 & x^2 & - & 4x & - & 5 & & \\
\hline
x-2 & x^3 & - & 6x^2 & + & 3x & + & 10 \\
 & \underline{x^3} & \underline{-} & \underline{2x^2} & & & & \\
 & & & -4x^2 & + & 3x & & \\
 & & & \underline{-4x^2} & \underline{+} & \underline{8x} & & \\
 & & & & & -5x & + & 10 \\
 & & & & & \underline{-5x} & \underline{+} & \underline{10} \\
 & & & & & & & 0
\end{array}
$$

So $(x^3 - 6x^2 + 3x + 10) \div (x - 2) = x^2 - 4x - 5$

and $x^3 - 6x^2 + 3x + 10 = (x - 2)(x^2 - 4x - 5)$

$= (x - 2)(x - 5)(x + 1)$

So $(x - 2)(x - 5)(x + 1) = 0$

and $x = 2$ or $x = 5$ or $x = -1$

Note that when dividing a polynomial like $x^5 + 2x^4 - 13x^2 + 10x$ with zero terms, it must be written with the zero terms included, like $x^5 + 2x^4 + 0x^3 - 13x^2 + 10x + 0$, so that there are columns for all terms in the answer.

Practice

Q7 If you have learnt the polynomial division algorithm, solve each of the following, first by factorising, then by graphing. If you haven't, just solve by graphing.

(a) $x^3 + 4x^2 + x - 6 = 0$

(b) $x^3 + 2x^2 - 5x = 6$

(c) $x^4 + 37x + 60 = 25x^2 + x^3$

(d) $x^5 + 2x^4 - 13x^2 + 10x = 0$

Some Applications of Polynomial Functions

Linear and quadratic functions are the most widely used polynomial functions. Constant functions are somewhat trivial and don't require a lot of processing.

Higher-degree polynomials tend to be used more in higher-level maths. Volumes are sometimes represented by cubic functions. The sum of a series of squares is a cubic function, the sum of a series of cubes is a quartic function and so on.

Many non-polynomial functions can be written as polynomials of infinite degree. For instance, a calculator works out cosines using the fact that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

[Note that 6! means 1×2×3×4×5×6 and that x has to be measured in radians. A radian is about 57°, $\frac{360}{2\pi}$ to be precise. Radians are the subject of Module M6-3.]

Solve

Q51 Find a sexual equation which has just 4 roots.

Q52 Find a cubic function with zeros of −2, 2 and 3. [Remember zeros are the x-values where it crosses the x-axis.]

Q53 Solve $(x + 2)(x - 3)(x^2 - 7x + 12) = 0$ without a graphics calculator.

Q54 Use the polynomial formula for $\cos x$ to find the cosine of 0.1 radians (about 5.7°) correct to 4 decimal places.

Revise

Revision Set 1

Q61 For each of the following functions, say whether it is a polynomial. If it isn't, say why. If it is, give its degree.

(a) $y = -x^3$

(b) $y = 4\frac{1}{2} + 5x^3 - x^4$

(c) $h = \sqrt{7}$

(d) $g = \frac{2}{x^3} + x^2$

(e) $r = 6c$

(f) $y = x + 3x^{2.5}$

(g) $h = \sqrt{3}(x + 3)(3x^2 - 5.5x)$

(h) $w = 10^a$

Q62 What are the names given to polynomial functions of degree 0 to 4?

Q63 (a) How many arms does a quartic function have?

(b) How many elbows does it have?

(c) What is the greatest number of turning points it can have?

(d) What is the greatest number of x-intercepts it can have?

(e) What is the smallest number of x-intercepts it can have?

Q64 Solve $0.2x^3 + x^2 - 4x - 7 = 5$

Revision Set 2

Q71 For each of the following functions, say whether it is a polynomial. If it isn't, say why. If it is, give its degree.

(a) $y = \frac{-1}{x}$

(b) $y = 2 + 0x^3 - 5x^4$

(c) $h = \sqrt{8}t$

(d) $g = \frac{2}{x^3} + 5x^2$

(e) $r = 2\sqrt{s}$

(f) $c = 2^a$

(g) $h = (x - 9)(4x^2 - 2.7\sqrt{x})$

(h) $y = x + x^{0.5}$

Q72 What are the names given to polynomial functions of degree 0 to 4?

Q73 (a) How many arms does a cubic function have?

(b) How many elbows does it have?

(c) What is the greatest number of turning points it can have?

(d) What is the greatest number of x-intercepts it can have?

(e) What is the smallest number of x-intercepts it can have?

Q74 Solve $x^5 - x^2 + 3 = 5$

Revision Set 3

Q81 For each of the following functions, say whether it is a polynomial. If it isn't, say why. If it is, give its degree.

(a) $y = x^5 - 3x^2$

(b) $y = 0$

(c) $p = x + \sqrt{3}$

(d) $g = 2^x$

(e) $r = \frac{2\sqrt{5}}{x^3}$

(f) $y = x + 3x^2 - 1$

(g) $h = \sqrt{2} \times 2x\,(3x^2 - 5.5x)$

(h) $y = 0.2^x$

Q82 What are the names given to polynomial functions of degree 0 to 5?

Q83 (a) How many arms does a quintic function have?

(b) How many elbows does it have?

(c) What is the greatest number of turning points it can have?

(d) What is the greatest number of x-intercepts it can have?

(e) What is the smallest number of x-intercepts it can have?

Q84 Solve $x^3 + 3x^2 + \sqrt{3}x - 7 = 0$

Answers

Q1 (a) yes, 3 (b) yes, 7
(c) yes, 1 (d) no, it has a negative power of f
(e) yes, 0 (f) yes, 2
(g) no, it has a fractional power of x (h) no, y is the square root of a polynomial
(i) yes, 12 (j) yes, 0
(k) yes, 2 (l) yes, 3

Q5 5

Q6 2, 0

Q7 (a) $x = 1, x = -2, x = -3$ (b) $x = 2, x = -1, x = -3$
(c) $x = -1, x = 3, x = 4, x = -5$ (d) $x = 0, x = 1, x = 1, x = -2, x = -5$

Q52 $y = (x + 2)(x - 2)(x - 3)$ or the same thing multiplied by a constant.

Q53 $x = -3, x = 3$, or $x = 4$ P4. 0.9950

Q61 (a) yes, 3 (b) yes, 4
(c) yes, 0 (d) no, it has a negative power of x
(e) yes, 1 (f) no, it has a fractional power of x
(g) yes, 3 (h) no, a is the exponent

Q62 0 constant; 1 linear; 2 quadratic; 3 cubic; 4 quartic

Q63 (a) 4 (b) 3 (c) 3 (d) 4 (e) 0

Q64 $x = -6.65, x = -2.29, x = 3.94$

Q71 (a) no, it has a negative power of x (b) yes, 4
(c) yes, 1 (d) no, it has a negative power of x
(e) no, it has a fractional power of x (f) no, a is the exponent
(g) no, it has a fractional power of x (h) no, it has a fractional power of x

Q72 0 constant; 1 linear; 2 quadratic; 3 cubic; 4 quartic

Q73 (a) 3 (b) 2 (c) 2 (d) 3 (e) 1

Q74 $x = -1.41, x = 1.41$

Q81 (a) yes, 5 (b) yes, 0
(c) yes, 1 (d) no, x is the exponent
(e) no, it has a negative power of x (f) yes, 2
(g) yes, 3 (h) no, x is the exponent

Q82 0 constant; 1 linear; 2 quadratic; 3 cubic; 4 quartic

Q83 (a) 5 (b) 4 (c) 4 (d) 5 (e) 1

Q84 $x = 1.11$