M1 Maths

A4-5 Algebraic Fractions

- manipulate algebraic fractions

Summary Learn Solve Revise Answers

Summary

There's not really anything new in this module. It's just some practice applying what you know about numerical fractions to algebraic fractions.

Learn

Equivalent Fraction Techniques

With number fractions you can multiply the top and the bottom by the same thing without changing the value of the fraction. Likewise with dividing.

For example, if you have $\frac{4}{10}$, you can multiply top and bottom by 3 to get $\frac{12}{30}$, or you can divide top and bottom by say 2 to get $\frac{2}{5}$. All these fractions are equivalent.

A rational expression is a fraction where the top and/or the bottom are algebraic expressions containing variables. Examples are $\frac{a}{2}$, $\frac{4x+2}{6x^2-5}$, $\frac{6m}{4m}$.

As with number fractions, you can multiply or divide the top and bottom by the same thing and the value doesn't change.

You can take $\frac{4a}{2}$ and multiply top and bottom by 5 to make $\frac{20a}{10}$ or by a to make $\frac{4a^2}{2a}$ or by x to make $\frac{4ax}{2x}$. Or you can divide by 2 to make $2a$. All these have the same value.

You can take $\frac{4x^3+2x}{6x^2-12x}$ and multiply the top and bottom by 3 to get $\frac{3(4x^3+2x)}{3(6x^2-12x)}$, which can be expanded to $\frac{12x^3+6x}{18x^2-36x}$. Or you can divide top and bottom by $2x$ to get $\frac{2x^2+1}{3x-6}$.

Remember, though, you must multiply or divide the whole top and the whole bottom, i.e. every term in the top and every term in the bottom.

You cannot cancel $\frac{4a+6}{8a+2}$ to $\frac{a+6}{2a+2}$ or to $\frac{4a+3}{8a+1}$. Doing this kind of thing is a common mistake in handling rational expressions.

Before dividing, it can help to factorise as this can make it obvious that the top and bottom have a common factor. For example, $\frac{4x^2+8x}{2x+4}$ can be factorised to $\frac{2x(2x+4)}{2x+4}$, which can be cancelled to $2x$.

Practice

Q1 For each of the following rational expressions, produce an equivalent expression by multiplying top and bottom by the expression given. Expand the result where possible.

(a) $\frac{m}{3}$ by 2

(b) $\frac{4x^2}{3}$ by x

(c) $\frac{4x^2}{3}$ by $3x$

(d) $\frac{a+5}{a}$ by 4

(e) $\frac{3}{7m}$ by $m+1$

(f) $\frac{4a^2+5}{8a-2}$ by $3a$

(g) $\frac{a^2-2a+5}{5a-2}$ by $3c$

(h) $\frac{4r}{r-3}$ by r

Q2 For each of the following rational expressions, produce an equivalent expression by dividing top and bottom by the expression given.

(a) $\frac{6n}{12}$ by 2

(b) $\frac{6n}{12}$ by 6

(c) $\frac{4p+10}{2p}$ by 2

(d) $\frac{3(h-1)}{6k}$ by 3

Q3 Cancel the following expressions as far as possible.

(a) $\frac{6}{9x}$

(b) $\frac{4t^2}{4t}$

(c) $\frac{4a+1}{4a}$

(d) $\frac{5(a+1)}{a+1}$

(e) $\frac{5e+10}{15e}$

(f) $\frac{v+10}{v+15}$

(g) $\frac{z-6}{3z-18}$

(h) $\frac{s}{s^3-8}$

(i) $\frac{3h(h-5)}{9h(h-5)}$

Combining and Separating Rational Expressions

You have learnt how to combine number fractions by adding, subtracting, multiplying and dividing them.

The same operations can be used to combine rational expressions. If you are rusty on adding, subtracting, multiplying and dividing fractions, go back and revise them (Module N2-7) before you go on.

Combining rational expressions by adding and subtracting

Suppose we wish to add or subtract $\frac{3a}{2}$ and $\frac{5a+1}{3a}$.

We multiply each expression top and bottom by the denominator of the other expression. This gives us $\frac{9a^2}{6a}$ and $\frac{10a+2}{6a}$.

Then we add or subtract the numerators, keeping the same denominator.

$$\frac{9a^2}{6a} + \frac{10a+2}{6a} = \frac{9a^2+10a+2}{6a} \qquad \frac{9a^2}{6a} - \frac{10a+2}{6a} = \frac{9a^2-10a-2}{6a}$$

It may be possible to collect terms or to cancel after combining. If it is possible, then this should generally be done.

Sometimes it is possible to spot a simpler common denominator. This can save time cancelling afterwards. For example

$$\frac{5a+3}{2a} + \frac{a+1}{3a} = \frac{15a+9}{6a} + \frac{2a+2}{6a} = \frac{17a+11}{6a}$$

Practice

Q4 Combine the following. Cancel and simplify if possible.

(a) $\frac{m}{3} + \frac{m}{4}$

(b) $\frac{2m+1}{3} - \frac{m-4}{2}$

(c) $\frac{2m+1}{2m} - \frac{m-4}{5}$

(d) $\frac{5}{3w} - \frac{4}{5}$

(e) $\frac{1-t}{3t} + \frac{t+2}{6t}$

(f) $\frac{4}{5} + \frac{m}{3}$

(g) $\frac{1}{x+1} - \frac{1}{x}$

(h) $\frac{1-u}{3-3u} + \frac{u+2}{u-1}$

Q5 Solve the following equations by first combining terms.

(a) $\frac{2r}{3} - \frac{r}{4} = 6$ (b) $\frac{d+3}{3} + \frac{2d-4}{5} = 0$

(c) $\frac{4}{c} - \frac{3}{2c} = \frac{3}{2}$ (d) $\frac{a^2+1}{3a} + \frac{5-2a^2}{6a} = 12$

Separating added and subtracted rational expressions

It is possible to rewrite $\frac{4}{7}$ as $\frac{2}{7} + \frac{2}{7}$ or as $\frac{3}{7} + \frac{1}{7}$ or even as $\frac{5}{7} - \frac{1}{7}$

The same can be done with algebraic rational expressions.

So $\frac{3b}{2b+5}$ $\frac{3b+1}{2b+5}$ can be separated into two terms: $\frac{3b}{2b+5} + \frac{1}{2b+5}$

$\frac{10a+2}{5a}$ can be separated into $\frac{10a}{5a} + \frac{2}{5a}$, which can then be cancelled to 2 + $\frac{2}{5a}$

Practice

Q6 Separate the following into two terms. Cancel and simplify if possible.

(a) $\frac{p+4}{3}$ (b) $\frac{2f+5}{3f}$

(c) $\frac{2s-1}{2s}$ (d) $\frac{2(a+3)}{3a^2}$

(e) $\frac{12t^2-6t}{3t}$ (f) $\frac{4b-2a}{2a}$

Combining rational expressions by multiplying

To multiply $\frac{2}{7}$ by $\frac{3}{5}$ we multiply the numerators to make the new numerator and likewise with the denominators.

So $\frac{2}{7} \times \frac{3}{5} = \frac{2 \times 3}{7 \times 5} = \frac{6}{35}$

We do exactly the same to multiply algebraic rational expressions.

So $\frac{4}{5} \times \frac{m}{3} = \frac{4m}{15}$; $\qquad \frac{1-t}{3t} \times \frac{t+2}{6t} = \frac{(1-t)(t+2)}{18t^2}$

Practice

Q7 Combine the following by multiplying. Cancel and simplify if possible.

(a) $\frac{m}{3} \times \frac{m}{4}$

(b) $\frac{2m+1}{3} \times \frac{2}{3m}$

(c) $\frac{2m+1}{m-4} \times \frac{m-4}{5}$

(d) $\frac{5}{3w} \times \frac{4w}{5}$

(e) $\frac{1-t}{3t} \times \frac{1}{6t}$

(f) $\frac{3}{e} \times \frac{e}{3}$

(g) $\frac{4}{2(x+1)} \times x$

(h) $\frac{1-u}{6+3u} \times \frac{u+2}{u-1}$

Separating multiplied rational expressions

If the numerator and denominator of a rational expression are products, then the expression can be separated into factors by the reverse process of multiplication. For instance

$\frac{3n(n+1)}{2(n+3)}$ can be separated into $\frac{3n}{2} \times \frac{(n+1)}{(n+3)}$

If only the numerator ***or*** denominator is a product, then the same can be done as follows.

$\frac{(n+1)}{2(n+3)} = \frac{1}{2} \times \frac{(n+1)}{(n+3)}$ or

$\frac{3n(n+1)}{(n+3)} = 3n \times \frac{(n+1)}{(n+3)}$

Of course, this can be done in different ways. The last example could be separated like this.

$\frac{3n(n+1)}{(n+3)} = \frac{3n}{(n+3)} \times (n+1)$

Practice

Q8 Separate the following rational expressions containing products.

(a) $\dfrac{p(p+4)}{3p}$

(b) $\dfrac{(2f+5)(f-1)}{3f+2}$

(c) $\dfrac{t^2}{2s}$

(d) $\dfrac{2(a+3)}{3a^2}$

(e) $\dfrac{2h^2}{3c}$

(f) $\dfrac{2}{a^3}$

Combining rational expressions by dividing

We divide by a number fraction by multiplying by the reciprocal of the fraction. The reciprocal of a fraction is the fraction turned upside down.

$$\frac{2}{5} \div \frac{3}{4} = \frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$$

We divide algebraic rational expressions exactly the same way. So

$$\frac{5}{3w} \div \frac{4w}{5} = \frac{5}{3w} \times \frac{5}{4w} = \frac{25}{12w^2}$$

Sometimes, when manipulating algebraic expressions, we can end up with a fraction over a fraction, like $\dfrac{12n}{\frac{4n}{5}}$.

This is the same as $12n \div \dfrac{4n}{5}$, which is $12n \times \dfrac{5}{4n}$, which is $\dfrac{60n}{4n}$, which is 15.

Practice

Q9 Combine the following by dividing. Cancel and simplify if possible.

(a) $\dfrac{m}{3} \div \dfrac{m}{4}$

(b) $\dfrac{2m+1}{3} \div \dfrac{2}{3m}$

(c) $\frac{2m+1}{m-4} \div \frac{m-4}{5}$

(d) $\frac{5}{3w} \div \frac{10}{4w}$

(e) $\frac{1-t}{3t} \div 6t$

(f) $\frac{3}{e} \div \frac{1}{2e}$

(g) $\frac{1}{2(x+1)} \div x$

(h) $\frac{1-u}{7+3u} \div \frac{u-1}{u+2}$

Solve

Q51 Simplify $\frac{x+1}{x+2} \times \frac{x+2}{x+3} \times \frac{x+3}{x+4} \times \frac{x+4}{x+5} \times \ldots \times \frac{x+11}{x+12} \times \frac{x+12}{x+13}$

Q52 If $\frac{2x+2}{4x} + \frac{x-a}{x} = \frac{3x-3}{2x}$, find a.

Q53 Simplify $\frac{x^2-2x-3}{x+1}$. [Hint: Factorise the top.]

Q54 Simplify $\frac{x^2-5x+6}{x^2+x-12}$

Q55 Solve $\frac{x^2-5x+2}{x^2+3x-20} = 2$

Revise

Revision Set 1

Q61 Cancel the following expressions as far as possible:

(a) $\frac{4x}{12x^2}$

(b) $\frac{t+2}{2t+2}$

Q62 Combine the following expressions:

(a) $\frac{x+1}{3x} + \frac{x}{6}$

(b) $\frac{2d+3}{d} \times \frac{3d}{d-4}$

(c) $3n \div \frac{n}{n-3}$

Q63 Separate $\frac{3b-4}{1-b}$ into two terms:

Q64 Write $\frac{x(2x+3)}{5}$ as the product of two factors.

Answers

Q1 (a) $\frac{2m}{6}$ (b) $\frac{4x^3}{6}$ (c) $\frac{12x^3}{9}$ (d) $\frac{4a+20}{4a}$

(e) $\frac{3m+3}{7m^2+7m}$ (f) $\frac{12a^3+15a}{24ma^2-6a}$ (g) $\frac{3a^2c-6ac+15c}{15ac-6c}$ (h) $\frac{4r^2}{r^2-3r}$

Q2 (a) $\frac{3n}{6}$ (b) $\frac{n}{2}$ (c) $\frac{2p+5}{p}$ (d) $\frac{h-1}{2k}$

Q3 (a) $\frac{2}{3x}$ (b) t (c) $\frac{4a+1}{4a}$

(d) 5 (e) $\frac{e+2}{3e}$ (f) $\frac{v+10}{v+15}$

(g) $\frac{1}{3}$ (h) $\frac{s}{s^3-8}$ (i) $\frac{1}{3}$

Q4 (a) $\frac{7m}{12}$ (b) $\frac{4m+11}{6}$ (c) $\frac{2m-2m^2+5}{10m}$ (d) $\frac{25-12w}{15w}$

(e) $\frac{4-t}{6t}$ (f) $\frac{12+5m}{15}$ (g) $\frac{1}{x^2+x}$ (h) $\frac{4u+5}{3u-3}$

Q5 (a) 14.4 (b) $-3/11$ (c) $5/3$ (d) $7/72$

Q6 (a) $\frac{p}{3}+\frac{4}{3}$ (b) $\frac{2}{3}+\frac{5}{3f}$ (c) $1-\frac{1}{2s}$ (d) $\frac{2}{3a}+\frac{2}{a^2}$

(e) $4t-2$ (f) $\frac{2b}{a}-1$

Q7 (a) $\frac{m^2}{12}$ (b) $\frac{4m+2}{9m}$ (c) $\frac{2m^2-7m-4}{5m-20}$ (d) $\frac{4}{3}$

(e) $\frac{1-t}{18t^2}$ (f) 1 (g) $\frac{4x}{2x+2}$ (h) $-\frac{1}{3}$

Q8 These can be done in various ways. Check your answer by multiplying.

Q9 (a) $\frac{4}{3}$ (b) $\frac{2m^2+m}{2}$ (c) $\frac{10m+5}{m^2-8m+16}$ (d) $\frac{2}{3}$

(e) $\frac{1-t}{18t^2}$ (f) 6 (g) $\frac{1}{2x^2+2x}$ (h) $-\frac{u+2}{7+3u}$

Q51 $\frac{x+1}{x+13}$ Q52 2 Q53 $x-3$ Q54 $\frac{x-2}{x+4}$ Q55 3, −14

Q61 (a) $\frac{1}{3x}$ (b) $\frac{t+2}{2t+2}$

Q62 (a) $\frac{2x^2+x+1}{3x}$ (b) $\frac{6d+9}{d-4}$ (c) $3n-9$

Q63 $\frac{3b}{1-b}-\frac{4}{1-b}$ Q64 $\frac{x}{5}\times(2x+3)$ or $x\times\frac{2x+3}{5}$