M1 Maths

A4-3 Simultaneous Equations – Linear

- solve pairs of linear simultaneous equations by equating, graphing, substitution and elimination

Summary Learn Solve Revise Answers

Summary

It is possible to solve equations with two unknowns, but we must have two equations and we solve them together, i.e. simultaneously.

There are several methods of solving simultaneous equations. In this module, we will learn four.

In **equating**, we rearrange the equations so that both have the same unknown as the subject. Then we equate the other sides of the equations and solve the resulting equation for the second unknown. Then we substitute the value of the second unknown into one of the original equations and solve to find the first unknown.

In **graphing**, we rearrange the equations as in the equating method, but then we graph both relations. The coordinates of the intersection are the solution to the equations.

In **substitution**, we rearrange just one of the equations and substitute it into the other. Then we solve that equation and continue as for the equating method.

In **elimination**, we add or subtract the two equations to get an equation with only one unknown, then continue as for equating. We may have to multiply one or both equations both sides by an integer to get one of the unknowns with the same or opposite coefficient so that it can be eliminated.

Learn

What are Simultaneous Equations?

Problem 1: At Drack's Pet Store, 3 bats cost $111. How much does one bat cost?

This problem can be solved by writing an equation. Admittedly, that's not the easiest way to solve it, but for the sake of illustration we will do it that way.

Let the price of a bat be b.

$b \times 3 = 111$

$\div 3 \quad \div 3$

$b = 37$

$\therefore$ the price of a bat is \$37.

Problem 2: At Drack's Pet Store, 3 bats cost \$111; 2 bats and a hoot owl cost \$94. How much does one bat cost? How much does one hoot owl cost?

This problem can be solved by writing two equations.

Let the price of a bat be b.

$b \times 3 = 111$

$\div 3 \rightarrow \quad b = 37$

$\therefore$ the price of a bat is \$37

Let the price of a hoot owl be h.

$h + 2 \times 37 = 94$

$h + 74 = 94$

$- 74 \rightarrow \quad h = 20$

$\therefore$ the price of a hoot owl is \$20

Problem 3: At Drack's Pet Store, 4 bats and 2 hoot owls cost \$188; 2 bats and 3 hoot owls cost \$134. How much does one bat cost? How much does one hoot owl cost?

This problem can be solved by writing two equations.

Let the price of a bat be b. Let the price of a hoot owl be h.

$4b + 2h = 188$

$2b + 3h = 134$

As with Problem 2, there are two unknowns and two equations. The difference is that neither of the equations can be solved by itself. There are ways to solve the equations, but they have to be solved together – simultaneously – they are simultaneous equations.

The reason why the equations in Problem 2 could be solved one at a time was that one of them had only one unknown in it: we could find the value of that unknown and then sub that value into the other equation. In Problem 3, both equations contain both unknowns. Because of this, neither can be solved by itself.

There are four main ways to solve simultaneous equations. They are:

1. equating,
2. graphing
3. substitution
4. elimination

You should learn all of these as not all methods can be used on all sets of simultaneous equations and different methods are easier for different sets.

Equating

Module A3-4 was about rearranging formulae in order to change the subject. Go and have a look at this now if you need a reminder on rearranging formulae.

To solve simultaneous equations by equating, we rearrange both equations so that they are explicit and so that the same unknown is the subject of both.

These are our equations.

$4b + 2h = 188$ . . . . . Eqn 1

$2b + 3h = 134$ . . . . . Eqn 2

We might choose b for our subject.

Start with Eqn 1.

$4b + 2h = 188$

$4b = 188 - 2h$

$b = 47 - \frac{1}{2}h$

Then do the same with Eqn 2.

$2b + 3h = 134$

$2b = 134 - 3h$

$b = 67 - 1\frac{1}{2}h$

We now know that $47 - \frac{1}{2}h$ is equal to b and that $67 - 1\frac{1}{2}h$ is equal to b.
Therefore $47 - \frac{1}{2}h$ must be equal to $67 - 1\frac{1}{2}h$

$47 - \frac{1}{2}h = 67 - 1\frac{1}{2}h$

We have equated the two expression for b, i.e. said that they are equal.

We now have an equation in one unknown and we can solve it. So we do.

$47 - \frac{1}{2}h = 67 - 1\frac{1}{2}h$

$47 = 67 - h$

$h + 47 = 67$

$h = 20$

So now we know a hoot owl costs \$20.

We can then substitute this value for h into either of the original equations to get an equation in just b. We will sub into Eqn 1.

$$4b + 2h = 188$$
$$4b + 2 \times 20 = 188$$
$$4b + 40 = 188$$
$$4b = 148$$
$$b = 37$$

So the price of a bat is \$37.

It is fairly easy to make mechanical errors when solving equations, so it is always worth checking our answers by substituting them into both original equations.

Eqn 1: $4b + 2h = 188$

$4 \times 37 + 2 \times 20 = 188$ This is correct.

Eqn 2: $2b + 3h = 134$

$2 \times 37 + 3 \times 20 = 134$ So is this.

So now we know our answers are correct.

Practice

Q1 For each of the following problems, write two simultaneous equations, then solve them by equating.

(a) 4 bats and 2 hoot owls cost \$112. 2 bats and 3 hoot owls cost \$108. Find the price of a bat and the price of a hoot owl.

(b) 1 bat and 3 hoot owls cost \$57. 3 bats and 2 hoot owls cost \$75. Find the price of a bat and the price of a hoot owl.

(c) 4 large jars and 2 small jars weigh 8.2 kg. 3 large jars and 3 small jars weigh 7.5 kg. Find the mass of a large jar and the mass of a small jar.

(d) It costs \$1960 to employ 4 carpenters and a labourer for a day. It costs \$2010 to employ 3 carpenters and 3 labourers for a day. How much does it cost to employ 2 carpenters and 4 labourers for a day?

Q2 Solve each of the following pairs of equations by equating.

(a) $4x + 2y = 22$ $3x + 4y = 24$

(b) $2a + 5b = 13$ $a + 2b = 8$

(c) $3n + 4p = 26$ $6n - p = 7$

(d) $c + f = 22.5$ $2c - 6f = 9$

(e) $3s - 4t = 8$ $4s - 2t = 19$

(f) $w - v = -11$ $5w + 2v = 14$

(g) $f - 4t = 8$ $\quad 2t - 5f = 7$

(h) $d - r = 6$ $\quad 2d - 4 = 3r$

(i) $a - 3b = -1.5$ $\quad 1.5 + 5b = 2a$

(j) $2s = 32$ $\quad 3s + t = 43$

Graphing

The first pair of equations we solved by equating was

$4b + 2h = 188$ . . . . . Eqn 1

$2b + 3h = 134$ . . . . . Eqn 2

We chose to make b the subject of both. This gave us

$b = 47 - ½h$

$b = 67 - 1½h$

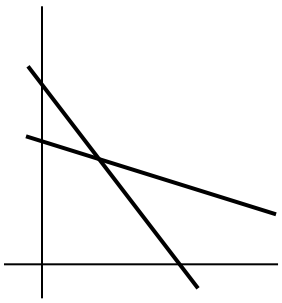
Using the graphing method, we would do exactly the same – up to this point. But then, instead of using algebra, we graph the two relations. This can be done by hand using what you know about linear functions, but that is laborious. A graphics calculator is a better idea. Use b as y and h as x. So we put in the two functions

Y1 = 47 – ½X

Y2 = 67 – 1½X

The graph of Y1 = 47 – ½X shows all the points where $b = 47 - ½h$

The graph of Y2 = 67 – 1½X shows all the points where $b = 67 - 1½h$



The intersection of the two lines is the only point where both statements are true. The coordinates of the intersection are the solution of the equation.

Think about this for a while until you can see why.

The coordinates of the intersection are (37, 20). So the solution is $b = 37$, $h = 20$.

Practice

Q3 For each of the questions in Q2, copy out the rearranged equations, then complete the solution by graphing.

Q4 Solve each of the following pairs of equations by graphing.

(a) $4x + 3y = 31$ $2x + 5y = 19$

(b) $2a + 3b = 8$ $a + 2b = 2$

(c) $4w + 2v = -4$ $-5w - 2v = 9$

(d) $4n + s = -23$ $n - s = -2$

(e) $2a - 3b = 6.5$ $1.5 + 5b = 2.5a - 1$

(f) $2s - t = 4$ $4s + t = 20$

Substitution

The first pair of equations we solved by equating and by graphing was

$4b + 2h = 188$ . . . . . Eqn 1

$2b + 3h = 134$ . . . . . Eqn 2

In each of the two methods, we rearranged the equations to make them explicit and to make b the subject of both. This gave us

$b = 47 - \frac{1}{2}h$

$b = 67 - 1\frac{1}{2}h$

With the substitution method, we only rearrange one of the equations. It doesn't matter which one. If we rearrange Eqn 1, we get

$b = 47 - \frac{1}{2}h$

We then substitute this expression for b into Eqn 2. In other words we replace b in Eqn 2 with $47 - \frac{1}{2}h$. So Eqn 2 becomes

$2(47 - \frac{1}{2}h) + 3h = 134$

This is now an equation in one unknown which we know how to solve.

$2(47 - \frac{1}{2}h) + 3h = 134$

$94 - h + 3h = 134$

$2h + 94 = 134$

$2h = 40$

$h = 20.$

Then, as with equating, we substitute $h = 20$ into either of the original equations to make an equation in b and solve it to find the value of b. If we sub into Eqn 1, we get.

$$4b + 2h = 188$$
$$4b + 2 \times 20 = 188$$
$$4b + 40 = 188$$
$$4b = 148$$
$$b = 37$$

So $h = 20$, $b = 37$

Practice

Q5 Solve each of the following pairs of equations by substitution.

(a) $5x + 2y = 34$ $\quad 3x + 4y = 26$

(b) $2a + 5b = 13$ $\quad 2b = 8 - a$

(c) $3n + 4p = 26$ $\quad 6n = 7 + p$

(d) $c + f = 2$ $\quad 2c - 5f = 11$

(e) $16t = 8 - s$ $\quad 5s = 19 + 4t$

(f) $4w - 2v = -28$ $\quad 5w + 2v = 1$

(g) $f - 4t - 8 = 0$ $\quad 2t - 5f = 7$

(h) $d + 4r = 31$ $\quad 2d + 15 = 3r$

(i) $a - 3b = 10$ $\quad 18 + 5b = 2a$

(j) $2s = 32$ $\quad 4s + t = 59$

Elimination

The elimination method is quite different. It is often the quickest and easiest method for linear equations.

The method involves adding or subtracting the two equations to get an equation in just one unknown, i.e. to eliminate the other unknown. For this to work, the unknown to be eliminated has to have the same or opposite co-efficient in the two equations.

Suppose we have the equations:

$$2x + 3y = 17 \quad \ldots\ldots\ldots \text{ Eqn 1}$$
$$2x - y = 9 \quad \ldots\ldots\ldots \text{ Eqn 2}$$

x has the same coefficient in both equations. If we subtract Equation 2 from Equation 1, we get

Eqn 2 – Eqn 1 →
$$\begin{aligned} 2x + 3y &= 17 \\ \underline{2x - y} &\underline{= 9} \\ 0x + 4y &= 8 \\ 4y &= 8 \end{aligned}$$

We can then solve this to get $y = 2$. Then we can sub this into either equation to find x, e.g.

$$\begin{aligned} 2x + 3 \times 2 &= 17 \\ 2x + 6 &= 17 \\ 2x &= 11 \\ x &= 5.5 \end{aligned}$$

So the solution is $x = 5.5$, $y = 2$.

If we have

$$\begin{aligned} x + 3y &= 15 \quad \ldots\ldots\ldots \text{ Eqn 1} \\ 5x - 3y &= 3 \quad \ldots\ldots\ldots \text{ Eqn 2} \end{aligned}$$

then we can add the two equations to get

Eqn 1 + Eqn 2 →
$$\begin{aligned} x + 3y &= 15 \\ \underline{5x - 3y} &\underline{= 3} \\ 6x + 0y &= 18 \\ 6x &= 18 \end{aligned}$$

Then we can solve this to get $x = 3$ and sub that into either equation to get $y = 4$.

Practice

Q6 Solve each of the following pairs of equations by elimination.

(a) $3x + 5y = 25$ $\quad 3x + 2y = 19$

(b) $2a - 5b = 3$ $\quad a + 5b = 9$

(c) $n + 4p = 26$ $\quad -n - p = -5$

(d) $c + f = 0$ $\quad c - 5f = 18$

(e) $3s - 4t = 18$ $\quad s - 4t = 6$

(f) $4m - 2k = 8$ $\quad 2k + 3m = 13$

Sometimes we won't have a common coefficient. But we can always get one by multiplying one or both equations by an integer.

For instance, if we have

$$\begin{aligned} 2x + 7y &= 48 \\ 4x - 3y &= 8 \end{aligned}$$

we can multiply the first equation by 2 to get $4x + 14y = 96$. Then we can subtract the two equations.

If we have

$$3x - 5y = 7$$
$$4x + 2y = 18$$

we can multiply the first equation by 4 and the second by 3 so we have $12x$ in both equations. The working might look like this.

$3x - 5y = 7$ . . . . . Eqn 1
$4x + 2y = 18$ . . . . . Eqn 2

$4 \times$ Eqn $1 - 3 \times$ Eqn $2 \rightarrow$

$$12x - 20y = 28$$
$$\underline{12x + 6y = 54}$$
$$-26y = -26$$
$$y = 1$$

Sub into Eqn 2 $\rightarrow$

$$4x + 2 = 18$$
$$x = 4$$

So the solution is $x = 4, y = 1$

Practice

Q7 Solve each of the following pairs of equations by elimination.

(a) $7s + 2t = 44$ $3s + 4t = 22$
(b) $3a + 5b = 13$ $-a + 2b = 2$
(c) $3n + 4p = 26$ $6n - p = 7$
(d) $f - c = 4$ $2c - 5f = 11$
(e) $3s - 4t = 14$ $4s - 3t = 21$
(f) $4w - 2v = -28$ $5w + 3v = 9$
(g) $f - 4t = 8$ $2t - 5f = 7$
(h) $7d + 4r = 49$ $2d + 15 = 3r$
(i) $a - 3b = -1.5$ $1.5 + 5b = 2a$
(j) $2s - 3t = 47$ $3s + t = 43$

Solving more than two simultaneous equations.

We can solve for more than two unknowns, but we need one equation for each unknown. So we could solve:

$$2x + 2y - z = -11$$
$$5x - 4y + z = 26$$
$$4x - 2y + 5z = 37$$

We will look at how to do this in Module A5-5.

Solve

Q51 If $a + b + c = 3$, $2a - 3b = 18$ and $-4c + 5 = 9$, find the value of $3a + 4b$.

Q52 Solve $a - 2b = 9$ and $a - b^2 = 1$.

Q53 Find another pair of equations with a square of one of the unknowns (like in Q52) with two integer solutions.

Revise

Revision Set 1

Q61 Solve the following by equating.
$3s + 5t = 24$
$3t - s = 20$

Q62 Solve the following by graphing.
$3b - 2c = 13$
$2b + 4c = 14$

Q63 Solve the following by substitution.
$3b - 4c = 10$
$2b + c = 5$

Q64 Solve the following by elimination.
$4x + 3y = 23$
$5x - 2y = 0$

Q65 Three adults and four children paid \$87 to get into the circus. On adult and two children paid \$41. How much are the entry costs for adults and children? Write equations and solve them to find the answers.

Revision Set 2

Q71 Solve the following by equating.
$4s + 5t = 22$
$2t - s = 14$

Q72 Solve the following by graphing.
$5b - 2c = 21$
$2b + 4c = 18$

Q73 Solve the following by substitution.

$4b - 4c = 12$

$2b + c = 5$

Q74 Solve the following by elimination.

$4x + 3y = 17$

$5x - 2y = 4$

Q75 Three pies and two sausage rolls cost \$42. On pie and three sausage rolls cost \$28. How much would one pie and one sausage roll cost? Write equations and solve them to find the answer.

Revision Set 3

Q81 Solve the following by equating.

$3p + 5r = 6$

$3p - r = -12$

Q82 Solve the following by graphing.

$3b - 2c = 7$

$2b + 4c = 26$

Q83 Solve the following by substitution.

$3b - 4c = 10$

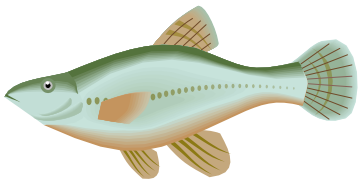
$5b + c = 9$

Q84 Solve the following by elimination.

$4x + 6y = 2$

$5x - 2y = 8$

Q85 Five loaves and two fishes weigh 4.8 kg. Two loaves and three fishes weigh 3.9 kg. How much will three loaves and four fished weigh? Write equations and solve them to find the answers.



Answers

Q1 (a) bat \$15, hoot owl \$26 (b) bat \$21, hoot owl \$12
(c) large 1.6 kg, small 0.9 kg (d) \$1820

Q2 (a) $x = 4, y = 3$ (b) $a = 14, b = -3$ (c) $n = 2, p = 5$ (d) $c = 18, f = 4.5$
(e) $s = 6, t = 2.5$ (f) $w = -2, v = 12$ (g) $f = -4, t = -3$ (h) $d = 14, r = 8$
(i) $a = 12, b = 4.5$ (j) $s = 16, t = -5$

Q3 (a) $x = 4, y = 3$ (b) $a = 14, b = -3$ (c) $n = 2, p = 5$ (d) $c = 18, f = 4.5$
(e) $s = 6, t = 2.5$ (f) $w = -2, v = 12$ (g) $f = -4, t = -3$ (h) $d = 14, r = 8$
(i) $a = 12, b = 4.5$ (j) $s = 16, t = -5$

Q4 (a) $x = 7, y = 1$ (b) $a = 10$, b $= -4$ (c) $w = -5, v = 8$
(d) $n = 4.025, s = 1.677$ or $n = -4.025, s = -1.677$
(e) $a = 10, b = 4.5$ (f) $s = 4, t = 4$

Q5 (a) $x = 6$, y $= 2$ (b) $a = 14, b = -3$ (c) $n = 2, p = 5$ (d) $c = 3, f = -1$
(e) $s = 4, t = 0.25$ (f) $w = -3, v = 8$ (g) $f = -4, t = -3$ (h) $d = 3, r = 7$
(i) $a = 4, b = -2$ (j) $s = 16, t = -5$

Q6 (a) $x = 5, y = 2$ (b) $a = 4, b = 1$ (c) $n = -2, p = 7$ (d) $c = 3, f = -3$
(e) $s = 6, t = 0$ (f) $m = 3, k = 2$

Q7 (a) $s = 6, t = 1$ (b) $a = 4, b = -3$ (c) $n = 2, p = 5$ (d) $f = 3, c = -1$
(e) $s = 6, t = 1$ (f) $w = -3, v = 8$ (g) $f = -4, t = -3$ (h) $d = 3, r = 7$
(i) $a = 12, b = 4.5$ (j) $s = 16, t = -5$

Q51 10 Q52 $a = 5, b = -2$ or $a = 17, b = 4$

Q61 $s = 1, t = 4$ Q62 $b = 5, c = 1$ Q63 $b = 2, c = -1$

Q64 $x = 2, y = 5$ Q65 adults \$17, children \$12

Q71 $s = -2, t = 6$ Q72 $b = 5, c = 2$ Q73 $b = 2, c = -1$

Q74 $x = 2, y = 3$ Q75 \$16

Q81 $p = -3, r = 3$ Q82 $b = 5, c = 4$ Q83 $b = 2, c = -1$

Q84 $x = 2, y = 1$ Q85 5.4 kg