M1 Maths

A3-7 Functions

- functions and the language of functions

Summary Learn Solve Revise Answers

Summary

A function is a relation in which, for every value of the independent variable, there is exactly one value of the dependent variable.

We say that there is a relation between two variables x and y; but we don't say that there is a function between the variables. Rather, we say that the dependent variable is a function of the independent variable.

A function can be thought of as a series of operations performed on the independent variable.

We often use the notation $y = f(x)$ to mean that the dependent variable (y) is a function of the independent variable (x). This can be pronounced 'y equals f of x'. If height, h, is a function of time, t, we can call height $h(t)$ to make this clear.

Learn

What is a function?

Relations are used most commonly to find the value of the dependent variable from the value of the independent variable.

Look at this relation between house number in Cactus Lane and the number of children living in the house.

House number	1	2	3	4	6	7	8	9	11	12	13	15	17
Number of children	0	4	2	0	0	3	1	2	0	0	2	0	3

House number is the independent variable. *Number of children* is the dependent variable. Given a value for the independent variable, it is always possible to give *the* value for the dependent variable. If the house number is 7, then the number of children is 3.

Now look at this relation between marks in the English test and Marks in the maths test for students in 10B.

English mark	6	7	4	6	9	5	7	8	7	7	2	6	6
Maths mark	9	7	8	7	8	8	10	9	6	7	0	5	8

Given a value for the independent variable in this relation, it is not always possible to give *the* value for the dependent variable. For instance, if the English mark is 6, then the maths mark could be 9, 7, 5 or 8.

There is something very convenient about the first relation (the children in the houses) compared to the second relation (the marks). We tend to use this type of relation most commonly and we give this type of relation a special name. It is called a **function**.

A function is a relation in which, for every value for the independent variable, there is exactly one value for the dependent variable.

The first relation above is a function. The second relation is not a function.

A function is sometimes called a *functional relation*. Unfortunately, there is no name in common use for a relation that is not a function. They can be called *dysfunctional relations*, but that name is not widely accepted by the mathematical community.

So some relations are functions and some are not. In fact, most of the relations that we have looked at up to now in our study of algebra have been functions and most of the relations you will look at in the rest of your time at school will be functions.

Note that the word function has a completely different meaning here from what it does in non-mathematical life. In non-mathematical life a function is a purpose or an event that someone puts on. The mathematical meaning has nothing to do with these everyday meanings.

Practice

Q1 Write four relations which are functions, one as a set of ordered pairs, one as a table, one as a graph and one as a formula.

Q2 Write four relation which are not functions, one as a set of ordered pairs, one as a table, one as a graph and one as a formula.

How to tell if a relation is functional or dysfunctional

As a set of ordered pairs

Which of the following two relations is a function?

Relation A: (3, 5), (8, 7), (6, 3), (5, 9), (6, 8)

Relation B: (3, 5), (8, 7), (6, 9), (5, 9), (7, 9)

If the relation is presented as a set of ordered pairs, then just look to see if there are any numbers that occur more than once as the independent variable (first number in the ordered pair). If there are, then the relation is dysfunctional. If not, then it is functional.

Relation A above is not a function because 6 occurs as the independent variable in more than one ordered pair, once with 3 as the dependent variable and once with 8 as the dependent variable.

Relation B above is a function because no number occurs more than once as the independent variable.

Note that in the last example, 9 occurred three times as the dependent variable. This does not stop the relation from being a function, because it is still true that, for any value for the independent variable, there is only one value for the dependent variable.

As a table

Which of the following two relations is a function?

Relation A:

House number	13	15	17	19
Number of pets	2	3	0	2

Relation B:

Number of pets	2	3	0	2
House number	13	15	17	19

If the relation is presented as a table, again just look to see if there are any numbers that occur more than once as the independent variable (the number in the top row or left column). If there are, then the relation is dysfunctional. If not, then it is functional.

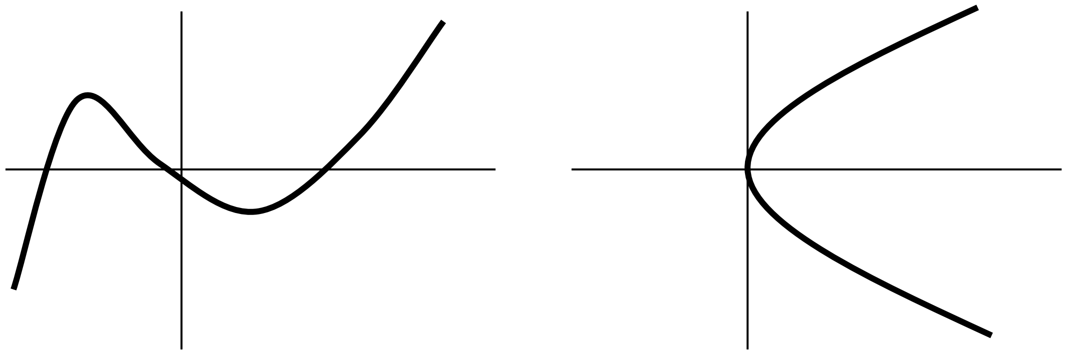
Relation A is a function because no number occurs more than once in the top row.

Relation B, however, is not a function because the value 2 occurs more than once in the top row.

You may have noticed that the two relations above are really the same information. The only difference is which variable is independent and which is dependent. Obviously, this can make a difference to whether the relation is a function.

As a graph

Which of the following two relations is a function?



It is easy to tell whether a relation is a function from its graph. If there are two points on the graph in the same vertical line, then that means that there are two ordered pairs with the same value for the independent variable. So the relation is not a function.

If no two points are in the same vertical line, then the relation is a function.

The graph on the left is a function. The one on the right is not.

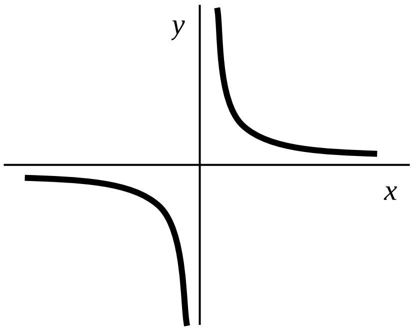
As a formula

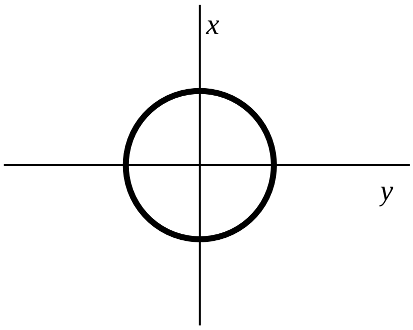
One way to tell if a formula is a function is to graph it. But there are other ways.

If a formula is of the form $y = \ldots\ldots$, where y is the dependent variable, then the relation will generally be a function. An exception is if the expression on the right contains a $\pm$ symbol. $\pm$ means plus or minus. So $y = \pm x$ means that, if $x = 5$ say, then $y = 5$ and -5. Whatever x is, y has two values, preventing that relation from being a function.

If a formula is not of the form $y = \ldots\ldots$, then it may or may not be a function. You can tell by thinking about it or by graphing it.

$xy = 4$ is a function (graph on left below), but $x^2 + y^2 = 4$ is not (graph on right below).





A y^2 in the formula will generally mean that the relation is not a function.

Practice

Q3 For each of the following relations, explain why it is or isn't a function.

(a) (1, 5), (2, 7), (3, 8), (4, 7), (5, 5)

(b) (6, 2), (8, 5), (3, 0), (8, –1), (7, 4)

(c)

p	10	20	30	40	50
z	5	11	14	11	8

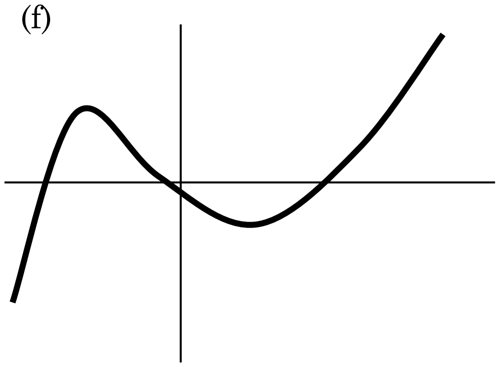
(d)

t	10	20	30	40	50
h	5	11	17	23	29

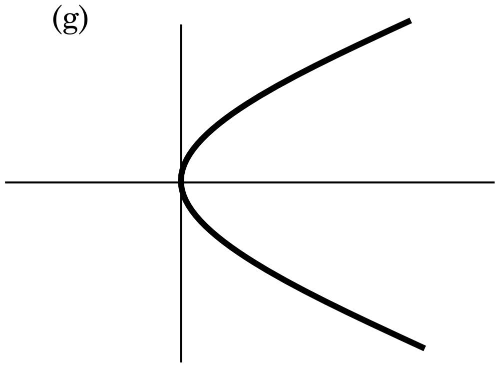
(e)

x	4	7	4	9	8
y	1	2	3	4	5

(f)



(g)



The Language of Functions

If a cannon ball is fired vertically upwards at 60 m/s, the following formula shows the height, h, in metres, at any time, t, in seconds, after firing.

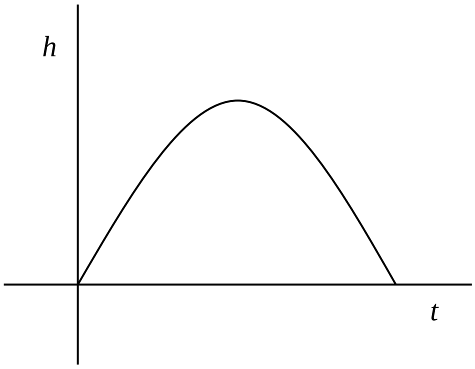
$h = 60t - 5t^2 \quad t \in \mathrm{R}, 0 \le t \le 12$

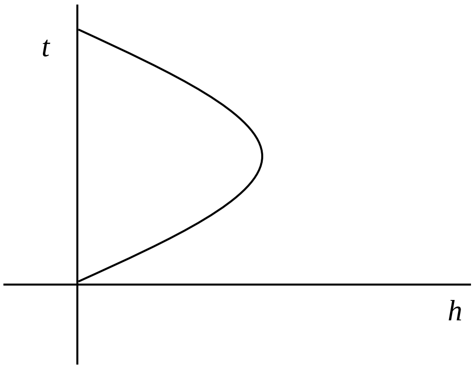
[Note that the domain is specified to show that the formula does not apply before $t = 0$, or after $t = 12$. The latter is because it will hit the ground again when $t = 12$.]

We say there is a relation between height and time. As this relation is a function, it might seem logical to say there is a function between height and time. But we don't. Why not?

The formula $h = 60t - 5t^2$ is a relation if t is the independent variable. But it is not a relation if h is the independent variable. Admittedly, h wouldn't normally be the independent variable if we wrote the formula as $h = 60t - 5t^2$, but it could be.

With t as the independent variable, the graph of the relation would be as shown on the left below. With h as the independent variable, the graph of the relation would be as shown on the right below.





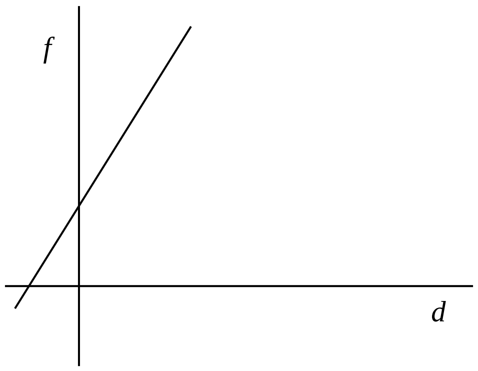
You will notice from the graphs that the relation as presented on the left is a function, whereas the relation as presented on the right is not. The one on the right is not a function because there are two values of the independent variable for some values of the dependent variable.

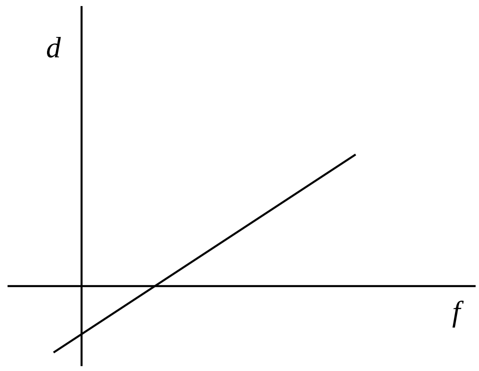
So the relation is a function one way, but not the other.

Because of this, we cannot say that there is a function between h and t. Instead we say h is a function of t, though t is not a function of h.

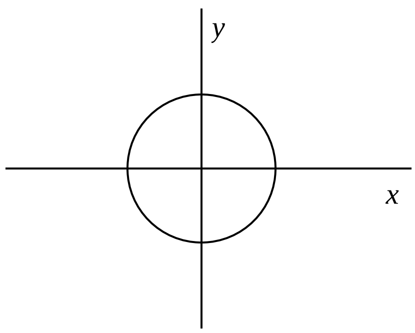
'h is a function of t' means that, as long as h is the dependent variable and t is the independent variable, then the relation is a function.

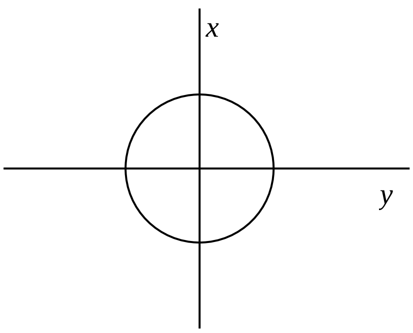
Some relations are a function both ways. If $f = 2d + 3$, f is a function of d and d is a function of f.





Other relations are not a function either way. $x^2 + y^2 = 4$ looks like this.





A Function as a Sequence of Operations

A function can also be thought of as something done to the independent variable:

the function $y = x + 2$ can be thought of as the function 'add 2';

the function $y = 5x$ can be thought of as the function 'multiply by 5';

the function $y = 5x + 2$ can be thought of as the function 'multiply by 5, then add 2';

the function $y = 3x^2 + 8$ can be thought of as the function 'square, then ×3, then +8'.

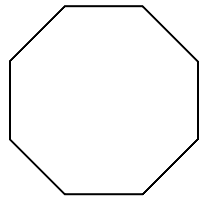
This is actually the way you will think about functions most. In this sense, a function is a sequence of operations. A function of x is a sequence of operations performed on x.

Function Notation

Each different sequence of operations is a different function. Adding 2 then multiplying by three can be written $(x + 2) \times 3$ . . . . if it is done to x. It would be called $(t + 2) \times 3$ if it was done to t. It would be called $(w + 2) \times 3$ if it was done to w and so on.

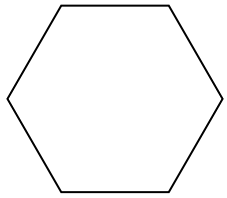
So functions can vary in terms of what the sequence of operations is and in terms of what variable the operations are performed on. We can name functions in a way that makes both these things clear.

Take the formula for the area of a regular octagon, $a = (3 + 4\sqrt{2})\, s^2$, where a is the area and s is the side length. We can call this function $a(s)$. We pronounce this 'a of s'. The a specifies the sequence of operations – in this case square, then multiply by $(3 + 4\sqrt{2})$; the s specifies what the operations are done to – in this case the side length, s.



So we say $a(s) = (3 + 4\sqrt{2})\, s^2$

The formula for the area of a regular hexagon would be $a = \frac{3\sqrt{3}}{2} s^2$



If we were talking about both these functions, to avoid confusion we might call the octagon function $o(s)$ and the hexagon function $h(s)$.

So $o(s) = (3 + 4\sqrt{2})\, s^2$ and $h(s) = \frac{3\sqrt{3}}{2}\, s^2$

Just like with variables, we can give a function any letter abbreviation we like, but we generally try to go for something that will remind us what it stands for.

If we wish to talk about a function such as 'multiply by 6 and add 1' without it having a particular context, we can give it any letter we like, e.g. $a(x)$, $k(x)$ etc., but it is customary to call it $f(x)$ (f standing for *function*). This is a bit like using x for the independent variable if there is no particular context.

f*(*x*)** is pronounced ***f* of *x

If we then want to talk about a second function, so as not to confuse it with $f(x)$, we call it $g(x)$. Then $h(x)$. We rarely need to go any further than that.

Let's say we define $f(x)$ as $x^2 + 5$. This means that f is the function 'square and add 5'. We can apply this function to numbers, like 4. Then we write $f(4) = 4^2 + 5$. In other words we do the function to 4. So $f(4) = 16 + 5 = 21$. In the same way $f(2) = 9$, $f(10) = 105$, $f(0) = 5$, $f(-4) = 21$ $f(½) = 5¼$ and so on.

Using expressions like $f(x)$, $f(2)$, $a(x)$, $h(t)$, $h(3)$ etc. is called function notation. It is a convenient way of writing statements about functions.

The notation *f*(*x*) means the sequence of operations, *f*, performed on *x*.

Practice

Q4 If $f(x) = 3x + 7$, find

(a) $f(2)$ (b) $f(11)$ (c) $f(0)$ (d) $f(-4)$

(e) $f(s)$ (f) $f(a+2)$ (g) $f(3p)$ (h) $f(x^2)$

Q5 If $g(x) = 2x^2 + 5$, find

(a) $g(2)$ (b) $g(11)$ (c) $g(0)$ (d) $g(-4)$

(e) $g(s)$ (f) $g(a+2)$ (g) $g(3p)$ (h) $g(x^2)$

Q6 If $y(x) = 2(x^2 + 1)$, find

(a) $y(2)$ (b) $y(11)$ (c) $y(0)$ (d) $y(-4)$

(e) $y(s)$ (f) $y(a+2)$ (g) $y(3p)$ (h) $y(x^2)$

Q7 If $p(a) = a^4$, find

(a) $p(2)$ (b) $p(11)$ (c) $p(0)$ (d) $p(-4)$

(e) $p(s)$ (f) $p(a+2)$ (g) $p(3t)$ (h) $p(x^2)$

Q8 If $a(r) = 2^r$, find

(a) $a(2)$ (b) $a(11)$ (c) $a(0)$ (d) $a(-4)$

(e) $a(s)$ (f) $a(b+2)$ (g) $a(3p)$ (h) $a(x^2)$

Q9. (a) $f(x) = 4x - 2$. Find x if $f(x) = 22$

(b) $f(c) = 10 - 5c$. Find c if $f(c) = 20$

(c) $t(r) = 5(r - 7)$. Find r if $t(r) = 45$

Solve

Q51 Consider the relation in which the independent variable is annual income of people in Mount Zargon and the dependent variable is the value of their house, would you expect this to be a function? Explain why or why not.

Q52 If $f(x) = x^2$ and $g(x) = 3x + 2$, find $f(g(5))$.

Q53 $f(c) = 31 - 5c^3$. Find c if $f(c) = -9$.

Revise

Revision Set 1

Q61 What is a function?

Q62 For each of the following relations, explain why it is or isn't a function.

(a)

y	3	7	6	4	3	8
x	11	5	7	3	4	8

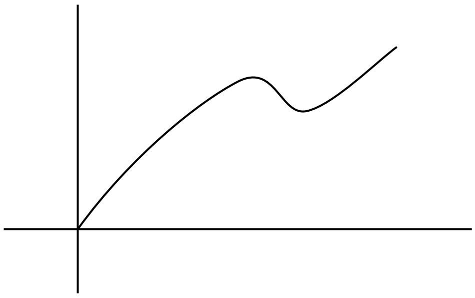
(b)

y	2	7	6	4	3	8
x	5	5	7	5	4	8

Q63 In the relation to the right,

(a) is h a function of t?

(b) is t a function of h?



Q64 If $p(x) = x^2 - 2$, find

(a) $p(3)$

(b) $p(2a)$

Answers

Q1 Answers will vary Q2 Answers will vary

Q3 (a) it is because for each value of the independent variable, there is just one value of the dependent variable

(b) it isn't because it contains the ordered pairs (8, 5) and (8, −1)

(c) it is because for each value of the independent variable, there is just one value of the dependent variable

(d) it is because for each value of the independent variable, there is just one value of the dependent variable

(e) it isn't because it contains the ordered pairs (4, 1) and (4, 3)

(f) it is because it would pass the vertical line test

(g) it isn't because it wouldn't pass the vertical line test

Q4 (a) 13 (b) 40 (c) 7 (d) −5

(e) $3s + 7$ (f) $3a + 13$ (g) $9p + 7$ (h) $3x^2 + 7$

Q5 (a) 13 (b) 247 (c) 5 (d) 37

(e) $2s^2 + 5$ (f) $2a^2 + 8a + 13$ (g) $18p^2 + 5$ (h) $2x^4 + 5$

Q6 (a) 10 (b) 244 (c) 2 (d) 34

(e) $2(s^2 + 1)$ (f) 2a2 + 8a + 10 (g) $18p^2 + 2$ (h) $2(x^4 + 1)$

Q7 (a) 16 (b) 11^4 (c) 0 (d) 256

(e) s^4 (f) $a^4 + 8a^3 + 24a^2 + 32a + 16$ (g) $81t^4$ (h) x^8

Q8 (a) 4 (b) 2048 (c) 1 (d) $^1/_{16}$

(e) 2^s (f) 4×2^b (g) 8^p (h) 2^{x^2}

Q9 (a) 6 (b) -2 (c) 16

Q51 No, because it is quite likely that two people will have the same job and the same pay, but live in different-value houses.

Q52 289 Q53 2

Q61 (a) a function is a relation in which, for every value for the independent variable, there is just one value for the dependent variable

Q62 (a) it isn't because it contains (3, 4) and (3, 11)

(b) it is because, for every value for the independent variable, there is just one value for the dependent variable.

Q63 (a) yes (b) no

Q64 (a) 7 (b) $2a^2 - 2$