M1 Maths

Learning by Thinking

A3-10 Index Laws 1-5

- using the index laws for natural number powers

Learn Answers

This LbT (Learning by Thinking) module is an alternative to the 'Learn' section of the normal module. It is designed to lead the student to work out the maths themselves by solving problems. Thus it contains only minimal explanations. The rationale behind the approach can be read **here**.

Learn

Recap of Powers

In Module N1-6 you learnt about powers. Here is a recap.

2^6, pronounced '2 to the power of 6' or '2 raised to the power of 6' or '2 to the 6th power' or '2 to the 6th', means 2×2×2×2×2×2, i.e. six 2s multiplied together. 2^6 is called a power; 2 is called the base and 6 is called the index or exponent. $2^6 = 64$,
$½^4 = ½×½×½×½ = 1/16$, $2.6^2 = 2.6×2.6 = 6.76$, $7^1 = 7$, $a^4 = a×a×a×a$,
$(-3)^2 = (-3)×(-3) = 9$.

One very common mistake with powers is to confuse -2^4 with $(-2)^4$. In the standard order of operations, raising to a power comes before subtraction or making something negative. So, in -2^4 we raise 2 to the power of 4 to get 16, then make it negative to get −16. However, in $(-2)^4$ the −2 is in brackets so it is done first, then −2 is raised to the power of 4 to get 16.

Q1 Evaluate without a calculator:

(a) 2^5	(b) 3^2	(c) 5^3	(d) 0^{12}	(e) 1^8
(f) 4^1	(g) 10^4	(h) $(½)^2$	(i) $(¼)^3$	(j) 9.2^1
(k) $(-4)^2$	(l) $(-3)^3$	(m) 0.3^2	(n) $(-½)^3$	(o) $(-0.1)^6$
(p) -2^2	(q) $(-2)^2$	(r) -2^3	(s) $(-2)^3$	(t) $(-3)^4$
(u) -3^4	(v) $-(½)^2$	(w) $(-½)^2$	(x) $(-10)^3$	(y) $(-10)^4$
(z) $-(-2)^5$				

Introduction to Laws 1-5

In Module A2-2, we learnt to simplify expressions by collecting terms. Remember, terms are things which are added together. In collecting terms, we put multiple terms together to make a single term: things like $5a + 2a = a+a+a+a+a + a+a = 7a$.

In this module we will learn to simplify expressions containing factors. Factors are things which are multiplied together. We will do things like collecting factors – things like $a^3 \times a^2 = a \times a \times a \times a \times a = a^5$.

Here, we will look at five simplification techniques. Each can be done by writing the powers in expanded form (e.g. a^4 as $a \times a \times a \times a$), performing the operation, then converting back to power form. But there are shortcuts which can make the job quicker. The questions are designed to lead you to these short cuts. The shortcuts are the first 5 of the 10 index laws. The other 5 index laws are dealt with in Module A5-2.

Index Law 1 - Multiplying Powers

Q2 Simplify the following expressions by changing to expanded form, then back to power form. For example, $t^3 \times t^2 = t \times t \times t \times t \times t = t^5$. If you find a short-cut method, use that instead.

(a) $t^4 \times t^2$ (b) $x^3 \times x^4$ (c) $s^3 \times s$ (d) $h^3 \times h^2$

(e) $n^2 \times n^2$ (f) $c \times c^2$ (g) $x^1 \times x^4$ (h) $a^5 \times a^2$

(i) $a^4 \times a^6$ (j) $k^8 \times k^7$ (k) $z^{12} \times z^{17}$ (l) $w^{45} \times w^{32}$

Hopefully, for part (l), you didn't write it out in expanded form, but rather, realised that the answer would be 45 ws and another 32 ws all multiplied together, i.e. 77 ws multiplied together, i.e. w^{77}.

In general, when we multiply two powers of the same variable, we just add the indices: $a^m \times a^n = a^{m+n}$. This is the first index law. Make sure this is obvious to you before you go on.

Q3 Simplify the following:

(a) $t^{14} \times t^{12}$ (b) $x^{30} \times x^{10}$ (c) $s^{52} \times s$ (d) $h^{125} \times h^{20}$

(e) $n^{48} \times n^{12}$ (f) $c \times c^{18}$ (g) $x^{34} \times x^{14}$ (h) $a^{27} \times a^{23}$

Q4 Simplify the following:

(a) $2t \times 3t^2$ (b) $5x^3 \times 2x^2$ (c) $s^6 \times 4s^2$ (d) $4h^3 \times 5h^2$

(e) $2n^2 \times -3n^2$ (f) $4c \times ½c^2$ (g) $7x^1 \times 2x^4$ (h) $-3a^5 \times -12a^2$

(i) $½a^4 \times ½a^6$ (j) $-k^5 \times 6k^7$ (k) $z^{12} \times 2z^7$ (l) $12w^{25} \times 4w^{13}$

(m) $7x^8 \times 2y^5$

Hopefully you realised that (m) can only be simplified to $14x^8y^5$. **Powers can only be combined if they have the same base**. $2^5 \times 3^2$ doesn't equal any simpler single power.

You should be able to work out how to do the following. If you're not sure, just write them out long-hand first.

Q5 Simplify the following as far as possible:

(a) $x^4 \times x^2 \times x^5$ (b) $x^2 \times x^1 \times x^4$ (c) $k^{13} \times k^{19} \times k^{17} \times \mathrm{k}^{11}$

(d) $h^6 \times p^2 \times h^4 \times p^3$ (e) $2n^5 \times a \times n^3$ (f) $4c^3 \times 5c^2 \times d^2 \times c^4$

(g) $2x^8 \times 2v^5 \times -x \times -3v^1 \times -5v^4$

Index Law 2 - Raising a Power to a Power

Q6 Simplify the following by expanding until you see the short-cut:

(a) $(t^3)^2$ (b) $(m^4)^2$ (c) $(s^2)^5$ (d) $(h^3)^1$ (e) $(a^{11})^3$

As you might have realised, to raise a power to a power, you can use the short-cut of multiplying the two powers.

This is the second index law: $(a^m)^n = a^{mn}$. Use this short-cut on the following.

Q7 Simplify:

(a) $(t^{12})^3$ (b) $(m^5)^{10}$ (c) $(s^6)^6$ (d) $(h^{15})^{10}$ (e) $(a^{11})^5$

In the following questions, you have to use the first and second laws. Don't forget that powers are done before multiplication, so with $(a^2)^4 \times a^5$, you simplify $(a^2)^4$ to a^8, then multiply by a^5 to get a^{13}.

Q8 Simplify:

(a) $(p^2)^3 \times p^5$ (b) $(t^4)^2 \times (t^2)^3$ (c) $(x^3)^2 \times x^5$

(d) $(k^2 \times k^3)^4$ (e) $(u^4 \times u^3)^2 \times u$ (f) $((p^2)^3 \times p^4)^2 \times (p^2)^5$

(g) $(k^6)^2 + 3(k^3)^4$ (h) $h^8 - (h^2)^4$ (i) $(w^2)^3 + w^5$

Hopefully you realised that in (g) and (h), you got like terms which could be added: $k^{12} + 3k^{12}$ to give $4k^{12}$ in (g) and $h^8 - h^8$ to give 0 in (h). In (i), the w^6 and w^5 are not like terms and so cannot be added: the expression must be left as $w^6 + w^5$.

We can only collect terms if they are **like terms**, i.e. they are the **same variable raised to the same power**.

Index Law 3 - Dividing Powers

Q9 Simplify the following. Write the divisions as fractions and cancel until you find the short-cut.

(a) $k^5 \div k^2$ (b) $a^7 \div a^3$ (c) $v^4 \div v^1$ (d) $k^{12} \div k^7$

(e) $d^{25} \div d^{22}$ (f) $4a^7 \div a^3$ (g) $6c^{28} \div 2c^{26}$ (h) $10a^{14} \div 5a^9$

(i) $5x^7 \div 12x^4$ (j) $t^6 \div h^3$ (k) $50k^3 \div 5d^4$ (l) $10a^9 \div 2w^9$

Hopefully, you realised, in doing (e), that the 22 ds on the bottom cancel 22 of the 25 ds on the top leaving just 3 ds on the top, thus giving an answer of d^3. Similarly with (g). In general, when you divide powers, you subtract the indices: $a^m \div a^n = a^{m-n}$. This is the third index law.

In the next question you will need to put all this together – multiplying powers, raising powers to powers and dividing.

Q10 Simplify:

(a) $\frac{h^2 \times h^3}{h}$ (b) $\frac{2a^3 \times a^5}{4a^4}$ (c) $\frac{4r^3 \times 2r^2}{24r^3 \times r}$

(d) $\frac{2(a^2)^4 \times a^3}{7a^3}$ (e) $(4s \times 2s^2) \div (12s^5)$ (f) $(4s \times 2s^2) \div (12s^5 \times 5s^3)$

(g) $3p^3 \times (p^5)^2 \div 9p^4$ (h) $\frac{(r^2r^3r)^4}{2r^5r^2}$ (i) $\frac{14s^2}{4s(s^2)^2}$

Handling More Than One Variable

Sometimes we need to simplify expressions containing more than one variable. We basically do what we've learnt above on each variable, but realise that we cannot combine different variables.

For example, to simplify $\frac{27a^3w^3 \times w^2}{12a^4w}$, we might use these steps:

$$\frac{27a^3w^3 \times w^2}{12a^4w}$$

Multiply the like variables $= \frac{27a^3w^5}{12a^4w}$

Cancel the like variables and numbers $= \frac{9w^4}{4a}$

Q11 Simplify as much as possible.

(a) $2x^2y \times 5xy^3$ (b) $\frac{12a^3b^2}{4a^4b}$ (c) $\frac{12rs^4 \times r^2ts}{18r^3st^5 \times 2rs^3}$

(d) $\frac{8av^4 \times v^2}{4a^3v}$ (e) $\frac{tv^4 \times 2vu^2}{2u^3vt^4}$ (f) $\frac{us^2}{3st^2}$

Index Law 4 – Powers of Products

Q12 Simplify as much as possible. Do it by expanding until you spot the short-cut.

(a) $(xy)^3$ (b) $(2ah)^3$ (c) $(2ap)^3 \times 3a^2$

(d) $\frac{(av)^4 \times v^2}{4a^3v}$ (e) $\frac{tv^4 \times (2vu)^2}{2(ut^2)^3v}$ (f) $\frac{s^2}{3st^2}$

(g) $\frac{(wu)^3 \times 5w^2}{u}$ (h) $\frac{(2r^2w^3h)^4}{4rw^{12}}$ (i) $\frac{(3rs)^4 \times r^2}{24r^3s}$

In general, $(abc\ldots)^n = a^nb^nc^n\ldots$ This is the fourth index law. Like the other index laws, this one should be fairly obvious because, for example,
$(abc)^4 = abcabcabcabc = aaaabbbbcccc = a^4b^4c^4$.

Index Law 5 – Powers of Fractions

Q13 Re-write without brackets and simplify if possible. Expand and multiply fractions until you spot the short-cut.

(a) $\left(\frac{a}{x}\right)^3$ (b) $\left(\frac{2}{s}\right)^4$ (c) $\left(\frac{w}{z}\right)^7$

(d) $\left(\frac{na}{h}\right)^3$ (e) $\left(\frac{2r}{s}\right)^2$ (f) $\left(\frac{3wv}{c}\right)^2$

(g) $\left(\frac{xy^2}{p}\right)^4$ (h) $\left(\frac{2t^3r}{4s^2}\right)^3$ (i) $\left(\frac{4ux^4}{2cx^2}\right)^2$

(j) $\left(\frac{(2rx)^3x^2}{8pr^2x}\right)^4$ (k) $\left(\frac{(4k^3r)^2 \times 3kr^2}{4k^2 \times (2kr)^2}\right)^3$

You should have discovered that $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. This is the 5th index law.

Summary of the 5 Index Laws

Below is a summary of the five laws.

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $(a^m)^n = a^{mn}$

Law 3: $a^m \div a^n = a^{m-n}$

Law 4: $(ab)^n = a^n b^n$

Law 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Number Applications of the Index Laws

The index laws can be useful in handling purely numerical applications as well. For instance, if we know that there are 10^{10} stars in an average galaxy and 10^{12} galaxies in the observable universe, then we can determine that there are $10^{10} \times 10^{12}$ stars in the universe. The first index law tells us that this is 10^{22} stars.

Q14 People tend to make mistakes with the following types of questions. Check that you get each one right before going on to the next. If you get any wrong, think about what you did in order to correct your misconception.

(a) There are about 10^{24} atoms in a gram of rock. The mass of the Earth is about 10^{28} grams. How many atoms in the Earth?

(b) A kilobyte is 2^{10} bytes. A megabyte is 2^{10} kilobytes. How many bytes in a megabyte. Answer as a power of 2 and as a numeral.

(c) There are 10^5 words in an average book, 10^4 books in an average library and 10^5 libraries in the world. How many words are there in the world's libraries?

(d) There are 10^{16} ants in the world. If an average ant colony consists of 10^5 ants, how many ant colonies are there?

(e) As you now know, there are 10^{16} ants in the world. If there were 10^{32}, would there be twice as many? Explain.

(f) A pea contains about 10^{22} molecules. How many molecules in 10 peas?

(g) How many molecules in 2 peas?

(h) A gram of hydrogen contains 6.02×10^{23} molecules (Avogadro's number). How many molecules in 2 tonnes of hydrogen. Give the answer in scientific notation. (1 t = 10^6 g)

(i) There are about 10^{81} charged particles in the observable universe. About half of these are electrons. How many electrons are there? Answer in scientific notation.

(j) What is the square of 10^6?

(k) What is the square root of 10^{40}?

(l) What is the cube root of 10^{24}?

(m) As I was going to Saint Ives I met a man with 10^4 wives. Each wife had 10^4 children, each child had 10^4 sacks. In each sack were 10^4 cats and each cat had 10^4 kittens. Show how to use the first or second index law to determine the number of kittens.

(n) A 1-litre cube is 10 cm wide. How wide is a 10^{15}-litre cube? Check your answer by calculating the volume of your cube in cm^3, then converting it to mL, then to litres.

(o) Find $8202^8 \div 4101^8$ without a calculator.

(p) Use a calculator to find the exact value of $(46 \times 36)^{15} \div (46 \times 18)^{15}$.

(q) Use a calculator to find $159^{51} \div 159^{49}$.

(r) Give $5^{42} \times 2^{42}$ in scientific notation.

(s) Solve $7^{24} = 49^x$.

Answers

Q1	(a) 32	(b) 9	(c) 125	(d) 0	(e) 1	
	(f) 7	(g) 10 000	(h) ¼	(i) 0.015625	(j) 4.2	
	(k) 16	(l) –27	(m) 0.09	(n) –0.125	(o) 0.000 001	
	(p) –4	(q) 4	(r) –8	(s) –8	(t) 81	
	(u) –81	(v) –¼	(w) ¼	(x) –1000	(y) 10 000	
	(z) 32					
Q2	(a) t^6	(b) x^7	(c) s^4	(d) h^5		
	(e) n^4	(f) c^3	(g) x^5	(h) a^7		
	(i) a^{10}	(j) k^{15}	(k) z^{29}	(l) w^{77}		
Q3	(a) t^{26}	(b) x^{40}	(c) s^{53}	(d) h^{145}		
	(e) n^{60}	(f) c^{19}	(g) x^{48}	(h) a^{50}		
Q4	(a) $6t^3$	(b) $10x^5$	(c) $4s^8$	(d) $20h^5$		
	(e) $-6n^4$	(f) $2c^3$	(g) $14x^5$	(h) $36a^7$		
	(i) $\frac{1}{4}a^{10}$	(j) $-6k^{12}$	(k) $2z^{19}$	(l) $48w^{38}$		
	(m) $14x^8y^5$					
Q5	(a) x^{11}	(b) x^7	(c) k^{60}	(d) $h^{10}p^5$	(e) $2an^8$	(f) $20c^9d^2$
	(g) $-60xv^{18}$					
Q6	(a) t^6	(b) m^8	(c) s^{36}	(d) h^{150}	(e) a^{55}	
Q7	(a) t^{36}	(b) m^{50}	(c) s^{10}	(d) h^3	(e) a^{33}	
Q8	(a) p^{11}	(b) t^{14}	(c) x	(d) k^{20}	(e) u^{15}	(f) p^{30}
	(g) $4k^{12}$	(h) 0	(i) $w^6 + w^5$			
Q9	(a) k^3	(b) a^4	(c) v^3	(d) k^5		
	(e) d^3	(f) $4a^4$	(g) $3c^2$	(h) $2a^5$		
	(i) $\frac{5}{12}x^3$	(j) $\frac{t^6}{h^3}$	(k) $\frac{10k^3}{d^4}$	(l) $\frac{5a^9}{w^9}$		
Q10	(a) h^4	(b) $\frac{a^4}{2}$	(c) $\frac{r}{3}$	(d) $\frac{2a^8}{7}$	(e) $\frac{2}{3s^2}$	(f) $\frac{2}{15s^5}$
	(g) $\frac{p^9}{3}$	(h) $\frac{1}{2}r^{17}$	(i) $\frac{7}{2s^3}$			
Q11	(a) $10x^3y^4$	(b) $\frac{3b}{a}$	(c) $\frac{s}{3rt^4}$	(d) $\frac{2v^5}{a^2}$	(e) $\frac{v^4}{ut^3}$	(f) $\frac{us}{3t^2}$
Q12	(a) x^3y^3	(b) $8a^3h^3$	(c) $24a^5p^3$	(d) $\frac{av^5}{4}$	(e) $\frac{2v^5}{ut^5}$	(f) $\frac{s}{3t^2}$
	(g) $5w^5u^2$	(h) $4r^7h^4$	(i) $\frac{27r^3s^3}{8}$			
Q13	(a) $\frac{a^3}{x^3}$	(b) $\frac{16}{s^4}$	(c) $\frac{w^7}{z^7}$	(d) $\frac{n^3a^3}{h^3}$	(e) $\frac{4r^2}{s^2}$	(f) $\frac{9w^2v^2}{c^2}$
	(g) $\frac{x^4y^8}{p^4}$	(h) $\frac{t^9r^3}{8s^6}$	(i) $\frac{8u^2x^4}{c^2}$	(j) $\frac{r^4x^{16}}{p^4}$	(k) $27k^9$	
Q14	(a) 10^{52}	(b) 2^{20}, 1 048 576		(c) 10^{14}	(d) 10^{11}	
	(e) No, there would be 10^{16} times as many			(f) 10^{23}	(g) 2×10^{22}	(h) 1.204×10^{30}
	(i) 5×10^{80}	(j) 10^{12}	(k) 10^{20}	(l) 10^8	(m) $(10^4)^5 = 10^{20}$	
	(n) 10 km	(o) $2^8 = 256$	(p) 32768	(q) 25281	(r) 1×10^{42}	(s) $x = 12$