M1 Maths

A3-10 Index Laws 1-5

- using the index laws for natural number powers

Summary   Learn   Solve   Revise   Answers

Summary

Expressions with powers can be simplified. We can do it by breaking them down into individual factors, e.g. $a^3 \times a^2 = a{\times}a{\times}a \times a{\times}a = a^5$. Or we can shortcut this method using 5 index laws. The 5 index laws are:

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $(a^m)^n = a^{mn}$

Law 3: $a^m \div a^n = a^{m-n}$

Law 4: $(ab)^n = a^n b^n$

Law 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Learn

Introduction

In Module A2-2, we learnt to simplify expressions by collecting terms. Remember terms are things which are added together. In collecting terms, we put multiple terms together to make a single term: things like $5a + 2a = 7a$.

In this module we will learn to simplify expressions containing factors. Factors are things which are multiplied together. We will do things like collecting factors – things like $a^2 \times a = a^3$.

There are five techniques. Each can be done by common sense – writing the powers in expanded form (e.g. a^4 as $a{\times}a{\times}a{\times}a$), then converting back to power form. But there are shortcuts which can make the job quicker. These shortcuts are the first 5 of the 10 index laws. The other 5 are dealt with in Module A5-2.

Recap of Powers

2^6, pronounced '2 to the power of 6' or '2 raised to the power of 6', means 2×2×2×2×2×2, i.e. six 2s multiplied together. 2^6 is called a power; 2 is called the base and 6 is called the index or exponent. $\frac{1}{2}^4 = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$, $2.6^2 = 2.6\times2.6$, $7^1 = 7$, $a^4 = a\times a\times a\times a$, $(-3)^2 = (-3)\times(-3) = 9$.

One very common mistake with powers is to confuse -2^4 with $(-2)^4$. In the standard order of operations, raising to a power comes before subtraction or making something negative. So, in -2^4 we raise 2 to the power of 4 to get 16, then make it negative to get −16. However, in $(-2)^4$ the −2 is in brackets so it is done first, then −2 is raised to the power of 4 to get 16.

Practice

Q1 Evaluate without a calculator:

(a) 2^5 (b) 3^2 (c) 5^3 (d) 0^{12} (e) 1^8

(f) 4^1 (g) 10^4 (h) $(\frac{1}{2})^2$ (i) $(\frac{1}{4})^3$ (j) 9.2^1

(k) $(-4)^2$ (l) $(-3)^3$ (m) 0.3^2 (n) $(-\frac{1}{2})^3$ (o) $(-0.1)^6$

(p) -2^2 (q) $(-2)^2$ (r) -2^3 (s) $(-2)^3$ (t) $(-3)^4$

(u) -3^4 (v) $-(\frac{1}{2})^2$ (w) $(-\frac{1}{2})^2$ (x) $(-10)^3$ (y) $(-10)^4$

(z) $-(-2)^5$

Index Law 1 - Multiplying Powers

We can multiply t^3 by t^2 like this:

$t^3 \times t^2 = t\times t\times t \times t\times t = t^5$.

We just write the powers out in expanded form (long-hand), then see how many factors there are.

Practice

Q2 Simplify the following:

(a) $t^4 \times t^2$ (b) $x^3 \times x^4$ (c) $s^3 \times s$ (d) $h^3 \times h^2$

(e) $n^2 \times n^2$ (f) $c \times c^2$ (g) $x^1 \times x^4$ (h) $a^5 \times a^2$

(i) $a^4 \times a^6$ (j) $k^8 \times k^7$ (k) $z^{12} \times z^{17}$ (l) $w^{45} \times w^{32}$

Hopefully, for the last question, you didn't write it out in expanded form, but rather, realised that the answer would be 45 *w*s and another 32 *w*s all multiplied together, i.e. 77 *w*s multiplied together, i.e. w^{77}.

In general, when we multiply two powers of the same variable, we just add the indices: $a^m \times a^n = a^{m+n}$. This is called the first index law. Make sure this is obvious to you before you go on. Use this idea to do Q3.

Practice

Q3 Simplify the following:

(a) $t^{14} \times t^{12}$ (b) $x^{30} \times x^{10}$ (c) $s^{52} \times s$ (d) $h^{125} \times h^{20}$

(e) $n^{48} \times n^{12}$ (f) $c \times c^{18}$ (g) $x^{34} \times x^{14}$ (h) $a^{27} \times a^{23}$

We can multiply $4t$ by $-3t^2$ like this:

$4t \times -3t^2 = 4 \times -3 \times t \times t^2 = 4 \times -3 \times t \times t \times t = -12t^3$.

Again, we write the expressions out long-hand, then rearrange the factors so that the numbers are together and the variables are together, then we multiply the numbers and re-write the variables as powers.

Practice

Q4 Simplify the following:

(a) $2t \times 3t^2$ (b) $5x^3 \times 2x^2$ (c) $s^6 \times 4s^2$ (d) $4h^3 \times 5h^2$

(e) $2n^2 \times -3n^2$ (f) $4c \times \frac{1}{2}c^2$ (g) $7x^1 \times 2x^4$ (h) $-3a^5 \times -12a^2$

(i) $\frac{1}{2}a^4 \times \frac{1}{2}a^6$ (j) $-k^5 \times 6k^7$ (k) $z^{12} \times 2z^7$ (l) $12w^{25} \times 4w^{13}$

(m) $7x^8 \times 2y^5$

Hopefully you realised that (m) can only be simplified to $14x^8y^5$. **Powers can only be combined if they have the same base**.

You should be able to work out how to do the following. If you're not sure, just write them out long-hand first.

Practice

Q5 Simplify the following as far as possible:

(a) $x^4 \times x^2 \times x^5$ (b) $x^2 \times x^1 \times x^4$ (c) $k^{13} \times k^{19} \times k^{17} \times \mathrm{k}^{11}$

(d) $h^6 \times p^2 \times h^4 \times p^3$ (e) $2n^5 \times a \times n^3$ (f) $4c^3 \times 5c^2 \times d^2 \times c^4$

(g) $2x^8 \times 2v^5 \times -x \times -3v^1 \times -5v^4$

Index Law 2 - Raising a Power to a Power

A power can be raised to a power. For example $(x^3)^4 = xxx \times xxx \times xxx \times xxx = x^{12}$

Practice

Q6 Simplify:

(a) $(t^3)^2$ (b) $(m^4)^2$ (c) $(s^2)^5$ (d) $(h^3)^1$ (e) $(a^{11})^3$

As you might have realised, to raise a power to a power, you can use the short-cut of multiplying the two powers.

This is the second index law: $(a^m)^n = a^{mn}$. Use this short-cut on the following.

Practice

Q7 Simplify:

(a) $(t^{12})^3$ (b) $(m^5)^{10}$ (c) $(s^6)^6$ (d) $(h^{15})^{10}$ (e) $(a^{11})^5$

In the following questions, you have to use the first and second laws. Don't forget that powers are done before multiplication, so with $(a^2)^4 \times a^5$, you simplify $(a^2)^4$ to a^8, then multiply by a^5 to get a^{13}.

Practice

Q8 Simplify:

(a) $(p^2)^3 \times p^5$ (b) $(t^4)^2 \times (t^2)^3$ (c) $(x^3)^2 \times x^5$

(d) $(k^2 \times k^3)^4$ (e) $(u^4 \times u^3)^2 \times u$ (f) $((p^2)^3 \times p^4)^2 \times (p^2)^5$

(g) $(k^6)^2 + 3(k^3)^4$ (h) $h^8 - (h^2)^4$ (i) $(w^2)^3 + w^5$

Hopefully you realised that in (g) and (h), you got like terms which could be added: $k^{12} + 3k^{12}$ to give $4k^{12}$ in (g) and $h^8 - h^8$ to give 0 in (h). In (i), the w^6 and w^5 are not like terms and so cannot be added: the expression must be left as $w^6 + w^5$.

We can only collect terms if they are **like terms**, i.e. they are the **same variable raised to the same power**.

Index Law 3 - Dividing Powers

When dividing powers, it is best to write them as the top and bottom of a fraction. For example, $4x^3 \div 12x^2$ should be written as $\frac{4x^3}{12x^2}$. This can then be written long-hand as $\frac{4xxx}{12xx}$.

Then we can do some cancelling: divide top and bottom by 4 and by xx to get $\frac{x}{3}$.

Practice

Q9 Simplify:

(a) $k^5 \div k^2$ (b) $a^7 \div a^3$ (c) $v^4 \div v^1$ (d) $k^{12} \div k^7$

(e) $d^{25} \div d^{22}$ (f) $4a^7 \div a^3$ (g) $6c^{28} \div 2c^{26}$ (h) $10a^{14} \div 5a^9$

(i) $5x^7 \div 12x^4$ (j) $t^6 \div h^3$ (k) $50k^3 \div 5d^4$ (l) $10a^9 \div 2w^9$

Hopefully, you realised, in doing (e), that the 22 ds on the bottom cancelled 22 of the 25 ds on the top leaving just 3 ds on the top, thus giving an answer of d^3. Similarly with (g). In general, when you divide powers, you subtract the indices: $a^m \div a^n = a^{m-n}$. This is the third index law.

In the next question you will need to put all this together – multiplying powers, raising powers to powers and dividing.

Practice

Q10 Simplify:

(a) $\frac{h^2 \times h^3}{h}$ (b) $\frac{2a^3 \times a^5}{4a^4}$ (c) $\frac{4r^3 \times 2r^2}{24r^3 \times r}$

(d) $\frac{2(a^2)^4 \times a^3}{7a^3}$ (e) $(4s \times 2s^2) \div (12s^5)$ (f) $(4s \times 2s^2) \div (12s^5 \times 5s^3)$

(g) $3p^3 \times (p^5)^2 \div 9p^4$ (h) $\frac{(r^2r^3r)^4}{2r^5r^2}$ (i) $\frac{14s^2}{4s(s^2)^2}$

Handling More Than One Variable

Sometimes we need to simplify expressions containing more than one variable. We basically do what we've learnt above on each variable, but realise that we cannot combine different variables.

For example, to simplify $\frac{27a^3w^3 \times w^2}{12a^4w}$, we might use these steps:

$$\frac{27a^3w^3 \times w^2}{12a^4w}$$

Multiply the like variables $= \frac{27a^3w^5}{12a^4w}$

Cancel the like variables and numbers $= \frac{9w^4}{4a}$

Practice

Q11 Simplify as much as possible.

(a) $2x^2y \times 5xy^3$ (b) $\frac{12a^3b^2}{4a^4b}$ (c) $\frac{12rs^4 \times r^2ts}{18r^3st^5 \times 2rs^3}$

(d) $\frac{8av^4 \times v^2}{4a^3v}$ (e) $\frac{tv^4 \times 2vu^2}{2u^3vt^4}$ (f) $\frac{us^2}{3st^2}$

Index Law 4 – Powers of Products

If we have two or more things multiplied together in a bracket and the whole thing raised to a power, then to expand, we need to raise everything inside the bracket to that power.

So for example $(2xy)^3 = 2xy \times 2xy \times 2xy = 2\times2\times2\times xxx \times yyy = 8x^3y^3$.

In general, $(abc\ldots)^n = a^nb^nc^n\ldots$ This is the fourth index law. Like the other index laws, this one should be fairly obvious because, for example,
$(abc)^4 = abcabcabcabc = aaaabbbbcccc = a^4b^4c^4$.

Practice

Q12 Simplify as much as possible. Don't forget that you need to expand brackets before you multiply or divide.

(a) $(xy)^3$ (b) $(2ah)^3$ (c) $(2ap)^3 \times 3a^2$

(d) $\frac{(av)^4 \times v^2}{4a^3v}$ (e) $\frac{tv^4 \times (2vu)^2}{2(ut^2)^3v}$ (f) $\frac{s^2}{3st^2}$

(g) $\frac{(wu)^3 \times 5w^2}{u}$ (h) $\frac{(2r^2w^3h)^4}{4rw^{12}}$ (i) $\frac{(3rs)^4 \times r^2}{24r^3s}$

Index Law 5 – Powers of Fractions

The last index law in this module is $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. This is just an application of multiplying fractions and should also be fairly obvious: just multiply all the tops together and multiply all the bottoms together. For example,

$\left(\frac{2}{5}\right)^3 = \frac{2}{5}\times\frac{2}{5}\times\frac{2}{5} = \frac{2\times2\times2}{5\times5\times5} = \frac{2^3}{5^3}$.

Practice

Q13 Re-write without brackets and simplify if possible:

(a) $\left(\frac{a}{x}\right)^3$ (b) $\left(\frac{2}{s}\right)^4$ (c) $\left(\frac{w}{z}\right)^7$

(d) $\left(\frac{na}{h}\right)^3$ (e) $\left(\frac{2r}{s}\right)^2$ (f) $\left(\frac{3wv}{c}\right)^2$

(g) $\left(\frac{xy^2}{p}\right)^4$ (h) $\left(\frac{2t^3r}{4s^2}\right)^3$ (i) $\left(\frac{4ux^4}{2cx^2}\right)^2$

(j) $\left(\frac{(2rx)^3x^2}{8pr^2x}\right)^4$ (k) $\left(\frac{(4k^3r)^2\times3kr^2}{4k^2\times(2kr)^2}\right)^3$

Summary of the 5 Index Laws

Below is a summary of the five laws in a form which you can use to make sure you know them. You should be able to write them out from memory. The order isn't crucial, but you should get them all.

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $(a^m)^n = a^{mn}$

Law 3: $a^m \div a^n = a^{m-n}$

Law 4: $(ab)^n = a^n b^n$

Law 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Number Applications of the Index Laws

The index laws can be useful in handling purely numerical applications as well. For instance, if we know that there are 10^{10} stars in an average galaxy and 10^{12} galaxies in the observable universe, then we can determine that there are $10^{10} \times 10^{12}$ stars in the universe. The first index law tells us that this is 10^{22} stars.

Practice

Q14 People tend to make mistakes with the following types of questions. Check that you get each one right before going on to the next. If you get any wrong, think about what you did in order to correct your misconception.

(a) There are about 10^{24} atoms in a gram of rock. The mass of the Earth is about 10^{28} grams. How many atoms in the Earth?

(b) A kilobyte is 2^{10} bytes. A megabyte is 2^{10} kilobytes. How many bytes in a megabyte. Answer as a power of 2 and as a numeral.

(c) There are 10^5 words in an average book, 10^4 books in an average library and 10^5 libraries in the world. How many words are there in the world's libraries?

(d) There are 10^{16} ants in the world. If an average ant colony consists of 10^5 ants, how many ant colonies are there?

(e) As you now know, there are 10^{16} ants in the world. If there were 10^{32}, would there be twice as many? Explain.

(f) A pea contains about 10^{22} molecules. How many molecules in 10 peas?

(g) How many molecules in 2 peas?

(h) A gram of hydrogen contains 6.02×10^{23} molecules (Avogadro's number). How many molecules in 2 tonnes of hydrogen. Give the answer in scientific notation. ($1 \text{ t} = 10^6 \text{ g}$)

(i) There are about 10^{81} charged particles in the observable universe. About half of these are electrons. How many electrons are there? Answer in scientific notation.

(j) What is the square of 10^6?

(k) What is the square root of 10^{40}?

(l) What is the cube root of 10^{24}?

(m) As I was going to Saint Ives I met a man with 10^4 wives. Each wife had 10^4 children, each child had 10^4 sacks. In each sack were 10^4 cats and each cat had 10^4 kittens. Show how to use the first or second index law to determine the number of kittens.

(n) A 1-litre cube is 10 cm wide. How wide is a 10^{15}-litre cube? Check your answer by calculating the volume of your cube in cm^3, then converting it to mL, then to litres.

(o) Use Index Law 5 to find $8202^8 \div 4101^8$ without a calculator.

(p) Use a calculator to find the exact value of $(46 \times 36)^{15} \div (46 \times 18)^{15}$.

(q) Use a calculator to find $159^{51} \div 159^{49}$.

(r) Give $5^{42} \times 2^{42}$ in scientific notation.

(s) Solve $7^{24} = 49^x$. Hint: use Law 2.

Solve

Q51 What is the last digit of 2^{100}? Hint: find the last digit of 2^1, 2^2, 2^3 and so on and look for a pattern.

Q52 What is the last digit of 7^{777}?

Q53 Simplify $\sqrt[7]{x^7y^{14}}$

Q54 Simplify $\sqrt[8]{x^4}$

Q55 $10^4 = 10\,000$; $10^3 = 1000$; $10^2 = 100$ . . . . In this pattern, the index is reduced by 1 each step; the power is divided by 10 each step. Continue the pattern to 10^{-4}.

Q56 Show how to use index laws to calculate $2^{39} \times (3^1/_3)^{22} \times 5^{18} \times 3^{24} \div 4^{11}$ without a calculator, giving the answer in scientific notation.

Revise

Revision Set 1

Q61 Write out Index Laws 1-5.

Q62 Evaluate:

(a) 3^3 (b) 7^1 (c) $(½)^3$ (d) $(-4)^2$ (e) -4^2 (f) $(-3)^3$

Q63 Simplify as far as possible:

(a) $a^4 \times a^2$ (b) $w^3 \times w$ (c) $3c^3 \times 5c \times d^2 \times 2c^4$ (d) $3p^4 \times -p^5 \times 4r^2$

(e) $(h^4)^3$ (f) $5a^4 \times b^3$ (g) $(v^2)^5 \times 3v^3$ (h) $a^8 \div a^3$

(i) $15x^7 \div 3x^2$ (j) $\frac{h^4 \times h^3}{h^2}$ (k) $\frac{(r^2r^7r)^3}{2r^4r^2}$ (l) $\frac{tv^2 \times 16vu^3}{8u^3vt^2}$

(m) $(2hy)^5$ (n) $4(ab^3)^2$ (o) $\frac{(3rs)^4 \times r^2}{24r^3s}$ (p) $\left(\frac{8t^2r}{4r^2}\right)^2$

Q64 Answer the following without a calculator:

(a) There are about 10^{10} stars in an average galaxy and 10^{12} galaxies in the universe. How many stars are there in the universe?

(b) If the stars of the universe together contain 10^{80} protons, how many protons are there in an average star?

(c) Find the exact value of $(62 \times 30)^{25} \div (31 \times 15)^{24} \div 2^{50}$.

Answers

Q1	(a) 32	(b) 9	(c) 125	(d) 0	(e) 1	
	(f) 7	(g) 10 000	(h) ¼	(i) 0.015625	(j) 4.2	
	(k) 16	(l) –27	(m) 0.09	(n) –0.125	(o) 0.000 001	
	(p) –4	(q) 4	(r) –8	(s) –8	(t) 81	
	(u) –81	(v) –¼	(w) ¼	(x) –1000	(y) 10 000	
	(z) 32					
Q2	(a) t^6	(b) x^7	(c) s^4	(d) h^5		
	(e) n^4	(f) c^3	(g) x^5	(h) a^7		
	(i) a^{10}	(j) k^{15}	(k) z^{29}	(l) w^{77}		
Q3	(a) t^{26}	(b) x^{40}	(c) s^{53}	(d) h^{145}		
	(e) n^{60}	(f) c^{19}	(g) x^{48}	(h) a^{50}		
Q4	(a) $6t^3$	(b) $10x^5$	(c) $4s^8$	(d) $20h^5$		
	(e) $-6n^4$	(f) $2c^3$	(g) $14x^5$	(h) $36a^7$		
	(i) $\frac{1}{4}a^{10}$	(j) $-6k^{12}$	(k) $2z^{19}$	(l) $48w^{38}$		
	(m) $14x^8y^5$					
Q5	(a) x^{11}	(b) x^7	(c) k^{60}	(d) $h^{10}p^5$	(e) $2an^8$	(f) $20c^9d^2$
	(g) $-60xv^{18}$					
Q6	(a) t^6	(b) m^8	(c) s^{36}	(d) h^{150}	(e) a^{55}	
Q7	(a) t^{36}	(b) m^{50}	(c) s^{10}	(d) h^3	(e) a^{33}	
Q8	(a) p^{11}	(b) t^{14}	(c) x	(d) k^{20}	(e) u^{15}	(f) p^{30}
	(g) $4k^{12}$	(h) 0	(i) $w^6 + w^5$			
Q9	(a) k^3	(b) a^4	(c) v^3	(d) k^5		
	(e) d^3	(f) $4a^4$	(g) $3c^2$	(h) $2a^5$		
	(i) $\frac{5}{12}x^3$	(j) $\frac{t^6}{h^3}$	(k) $\frac{10k^3}{d^4}$	(l) $\frac{5a^9}{w^9}$		
Q10	(a) h^4	(b) $\frac{a^4}{2}$	(c) $\frac{r}{3}$	(d) $\frac{2a^8}{7}$	(e) $\frac{2}{3s^2}$	(f) $\frac{2}{15s^5}$
	(g) $\frac{p^9}{3}$	(h) $\frac{1}{2}r^{17}$	(i) $\frac{7}{2s^3}$			
Q11	(a) $10x^3y^4$	(b) $\frac{3b}{a}$	(c) $\frac{s}{3rt^4}$	(d) $\frac{2v^5}{a^2}$	(e) $\frac{v^4}{ut^3}$	(f) $\frac{us}{3t^2}$
Q12	(a) x^3y^3	(b) $8a^3h^3$	(c) $24a^5p^3$	(d) $\frac{av^5}{4}$	(e) $\frac{2v^5}{ut^5}$	(f) $\frac{s}{3t^2}$
	(g) $5w^5u^2$	(h) $4r^7h^4$	(i) $\frac{27r^3s^3}{8}$			
Q13	(a) $\frac{a^3}{x^3}$	(b) $\frac{16}{s^4}$	(c) $\frac{w^7}{z^7}$	(d) $\frac{n^3a^3}{h^3}$	(e) $\frac{4r^2}{s^2}$	(f) $\frac{9w^2v^2}{c^2}$
	(g) $\frac{x^4y^8}{p^4}$	(h) $\frac{t^9r^3}{8s^6}$	(i) $\frac{8u^2x^4}{c^2}$	(j) $\frac{r^4x^{16}}{p^4}$	(k) $27k^9$	
Q14	(a) 10^{52}	(b) 2^{20}, 1 048 576		(c) 10^{14}	(d) 10^{11}	
	(e) No, there would be 10^{16} times as many			(f) 10^{23}	(g) 2×10^{22}	(h) 1.204×10^{30}
	(i) 5×10^{80}	(j) 10^{12}	(k) 10^{20}	(l) 10^8	(m) $(10^4)^5 = 10^{20}$	
	(n) 10 km	(o) $2^8 = 256$	(p) 32768	(q) 25281	(r) 1×10^{42}	(s) 12

Q51 6   Q52 7   Q53 xy^2 Q54 $\sqrt{x}$ Q55 $10^{-4} = 0.0001$ Q56 4.5×10^{40}

Q61	See text					
Q62	(a) 27	(b) 7	(c) $\frac{1}{8}$	(d) 16	(e) −16	(f) −27

Q63	(a) a^8	(b) w^4	(c) $30c^8d^2$	(d) $-12p^9r^2$
	(e) h^{12}	(f) $5a^4b^3$	(g) $3v^{13}$	(h) a^5
	(i) $5x^5$	(j) h^5	(k) $\frac{r^{24}}{2}$	(l) $\frac{2v^2}{t}$
	(m) $32h^5y^5$	(n) $4a^2b^6$	(o) $\frac{27r^3s^3}{8}$	(p) $\frac{4t^4}{r^2}$